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NUOVO CIMENTO

A CURA DELLA SOCIETÀ ITALIANA DI FISICA

1959 4° Trimestre N. 2

RENDICONTI

DEL

IX CORSO CHE NELLA VILLA MONASTERO A VARENNA

DAL 18 AL 30 AGOSTO 1958

FU TENUTO A CURA

DELLA SCUOLA INTERNAZIONALE DI FISICA

DELLA SOCIETÀ ITALIANA DI FISICA

SULLA

FISICA DEI PIONI

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La Direzione del Nuovo Cimento *porge al prof.* B. Touschek *il più vivo ringraziamento per tutto il lavoro svolto nella raccolta e nella sistemazione dei testi definitivi per la stampa di questo fascicolo.*

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INTRODUZIONE

Parole inaugurali

DI

G. Polvani

Presidente della Società Italiana di Fisica

È questa la quarta volta che nella sessione estiva 1958 della Scuola Internazionale di Fisica della nostra Società, io salgo a questo banco per ripetere un discorso che, più o meno, è sempre il medesimo, almeno nelle sue linee generali.

Ma questo che adesso inizio contiene un elemento nuovo, del quale veramente parlerò più avanti, esaurita la parte tradizionale propria di questi miei interventi inaugurativi, ma al quale ho voluto già accennare fin dall'inizio, per rinfrancar me stesso della quadruplice fatica e suscitare in quelli che mi ascoltano e già mi hanno sentito parlare le altre tre volte una qualche nuova attesa e un qualche maggiore interesse.

Esauriamo dunque dapprima la parte tradizionale. Questa si compie porgendo anzitutto il saluto della Società Italiana di Fisica a tutti i convenuti, e specie a coloro che in un modo o in un altro parteciperanno attivamente al Corso. Ben cinquantun allievo, di contro ai quaranta annunciati nel bando — al solito le domande sono state assai numerose e come sempre il giuoco del chiedere e del concedere si è chiuso, per la Direzione della Scuola, con la insoddisfazione di non poter far di più —; sei docenti, cui si aggiungeranno altri fisici incaricati di trattare in seminari e discussioni punti particolari degli argomenti che fanno l'oggetto del Corso; infine una decina di uditori interessati a frequentare il Corso per alcune particolari questioni: in totale dunque una settantina di persone. La «forza» — per adoperare un termine militare – di questo IV Corso della sessione estiva 1958 non è dunque davvero minore di quelli dei corsi che l'hanno preceduto.

Come sempre presenterò reciprocamente insegnanti, allievi e uditori.

Insegnanti: Julius Ashkin di Pittsburg, Gilberto Bernardini di Genève, James MacDonald Cassels di Liverpool, Marcello Cini di Roma, Ernesto Corinaldesi di Keele, Bernard Peters di Kopenhagen, Walter Thirring di Bern, Bruno Touschek di Roma.

Allievi: W. Alles di Bologna, A. Astbury di Liverpool, A. Barbaro di Roma, B. Barsella di Pisa, G. Bellettini di Roma, L. Bertanza di Pisa, M. Bloch di Paris, V. Borelli di Bologna, B. Brisson Fouché di Paris, A. Byfield di New York City, N. Burgoyne di Kopenhagen, N. Cabibbo di Roma, A. Citron di Genève, G. Culligan di Liverpool, B. De Tollis di Roma E. Eberle di Catania, G. Eder di Wien, E. Ferrari di Roma, H. Filthuth di Genève, G. Finocchiaro di Roma, E. Fiorini di Milano, P. Franzini di Pisa, B. Frisken di Birmingham, R. Gessaroli di Bologna, M. Grilli di Padova, L. Halpern di Wien, R. Hanna di Genève, P. Hillman di Genève, W. Hogg di Glasow, D. Holthuizen di Amsterdam, M. Huybrecht di Bruxelles, B. Jongjeans di Amsterdam, M. Kemp di Liverpool, D. Kleitman di Kopenhagen, J. Knight di Maryland, W. Koch di Bern, G. Leotta di Bologna, L. Lovitch di Cambridge, J. Major di Durham, I. Mannelli di Pisa, T. Massam di Liverpool, W. Menhardt di Göttingen, W. Middelkoop di Geneve, G. Moneti di Roma, H. Paul di Genève, E. Perez Ferreira di Durham, J. Reuss di Leiden, R. Santangelo di Padova, V. Silvestrini di Pisa, R. Stora di Saclay, B. Ugdaonkar di Bombay, S. Vitale di Genova, P. Waloschek di Bologna, A. Zichichi di Genève.

Uditori: G. Baroni di Roma, M. Beneventano di Roma, P. Falk-Vairant di Saclay, J. Gaillard di Saclay, F. Kortel di Istanbul, A. Levèque di di Saclay, L. Mezzetti di Roma, R. Sherr di Princeton, L. Van Rossum di Saclay.

Dopo i saluti e le presentazioni, i ringraziamenti: anzitutto ai professori per la loro collaborazione a questa Scuola nel comunicare agli altri il loro prezioso sapere perchè fruttifichi e giovi all'ampliamento della nostra Scienza.

Un ringraziamento particolare poi al prof. Touscheck per la sua opera di organizzatore del Corso e di Direttore, quest'ultima carica da oggi particolarmente gravosa e difficile.

Un rinnovato ringraziamento infine a tutti coloro — enti locali, provinciali, nazionali, società e fondazioni italiane ed estere, e privati — che in un modo o in un altro hanno dato il loro aiuto materiale e morale per il funzionamento di questa Scuola per l'anno in corso 1958. Già li ho nominati all'inaugurazione del primo Corso: l'omissione ora dei loro nomi — spero concessa per amore di brevità — non significa affatto che essi ci siano oggi meno presenti alla mente e al cuore: tutt'altro. Il loro aiuto è anzi giorno per giorno sempre più apprezzato e la riconoscenza aumentata.

Delle questioni che il Corso tratterà, del tono che il Corso stesso avrà, della sua impostazione..., vi parlerà con la sua competenza specifica il prof. Touschek. Qui io potrò tutt'al più fare notare come il Corso in un qualche modo si riconnetta con i due primi qui svolti nel 1953 e 1954 riprendendo, sotto particolari aspetti e con le conoscenze acquisite in cinque anni di studi, a trat-

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tare di quelle particelle, i mesoni, che tanta importanza hanno nei processi nucleari di media-alta energia.

Con questi brevi accenni la parte tradizionale comune del mio discorso potrebbe esser terminata e potrei ormai pronunciare la consueta formula inaugurale.

Ma ho premesso che un qualche cosa di nuovo vorrei aggiungere: una breve considerazione relativa al consuntivo di questa sessione estiva 1958.

La considerazione è la seguente: che quest'anno di contro ai diciannove giorni lavorativi dei Corsi svolti in ciascuno dei quattro anni precedenti (tralasciando quello del 1953 che fu di venticinque giorni), il numero totale di giorni lavorativi della nostra Scuola Internazionale di Fisica, è salito a tre volte tanto quello degli anni precedenti; cioè a sessanta giorni, divisi in quattro Corsi che hanno toccato argomenti profondamente diversi: la Fisica del plasma, la Teoria dell'informazione, i Problemi matematici della Teoria quantistica delle particelle e dei campi, la Fisica dei pioni.

E quale è la ragione di questa molteplicità di argomenti e di questo aumento del numero dei corsi? Accresciuto interessamento alla nostra Scuola? Rapido ampliamento della nostra Scienza? Maggiore interessamento in Italia alle questioni di Fisica?

Probabilmente ognuna di queste cause ha influito. Certo è che la Società Italiana di Fisica ampliando così notevolmente la sua attività varennina, crede — è sicura anzi — di aver corrisposto ad un bisogno internazionale. Ne fa fede il numero notevole delle domande di frequenza ai corsi. Ne fa fede il fatto che già ci domandano cosa faremo l'anno prossimo o ci suggeriscono cosa fare.

La realtà è che — per esprimermi in termini matematici — la derivata della fortuna rispetto al tempo, di questa nostra Scuola è decisamente positiva. Sta a tutti coloro — Società Italiana di Fisica, Enti nazionali, provinciali, locali, privati — che la fanno vivere, di aiutarla portandola verso un ancor maggiore sviluppo, soprattutto verso una maggiore sicurezza di vita.

Forse il discorso se inoltrato per questa via, troppo dilagherebbe ed esorbiterebbe da quel ristretto accenno cui deve esser qui contenuto. Ma chi ha della Scuola la responsabilità — responsabilità non solo, dirò, italiana, nazionale, ma anche internazionale — deve guardare assai più al di là di quello che è il limite annuale, l'episodio annuale. Ora di questo più lontano sguardo, oggi soprattutto è il momento. Il non essere consapevole o il tacerlo sarebbe colpevolezza. Ma basti qui l'accenno, e ritorniamo al Corso che oggi s'inizia.

Con l'augurio che anche ad esso arrida il felice successo che gli altri ha coronato, dichiaro aperto il IV Corso 1958 — IX dall'inizio della Scuola Internazionale di Fisica della nostra Società — dedicato alla Fisica dei pioni.

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SUPPLEMENTO AL VOLUME XIV, SERIE X
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La Fisica dei pioni.

PROLUSIONE

di

B. TOUSCHEK

Direttore del Corso

Nello sviluppo della Fisica dei mesoni si possono distinguere tre periodi. Il primo si iniziò quando nel 1935 il grande scienziato giapponese YUKAWA ebbe l'idea di connettere il corto raggio d'azione delle forze nucleari con la massa di una ipotetica particella, il mesone. Questo primo periodo era soprattutto caratterizzato dalla falsa identificazione della particella ideata da YUKAWA con certe particelle osservate nella radiazione cosmica: i mesoni di Anderson. Il secondo periodo nello sviluppo della Fisica dei mesoni si iniziò con la scoperta di LATTES, OCCHIALINI e POWELL di un mesone di massa uguale a circa 280 masse elettroniche, chiamato mesone π (identificabile con la particella di Yukawa), che decadendo produceva un altro mesone identificabile con quello di Anderson. In questo periodo fu determinante la comparsa delle prime macchine acceleratrici capaci di produrre particelle aventi energie intorno ai 300 milioni di elettron-volt, sufficienti per la produzione artificiale dei mesoni π, ovvero dei pioni. Le esperienze fatte con i mesoni prodotti artificialmente portarono a stabilire le conclusioni seguenti:

1) I mesoni π sono pseudoscalari (e non vettoriali).

2) È confermata l'esistenza di tre stati di carica, positiva, negativa e nulla. Il mesone neutro decadein due fotoni di vita media molto breve, ed è responsabile in gran parte della generazione di cascate nella radiazione cosmica.

3) Esiste un'interazione fortissima tra mesoni e nucleoni in uno stato (il cosiddetto $P_{\frac{3}{2}\frac{3}{2}}$) di moto relativo fra nucleone e mesone, che conduce a una risonanza nella diffusione dei mesoni a un'energia di circa 200 milioni di elettron-volt. Lo stato di questo ramo della Fisica fu chiaramente discusso

nel Corso di Varenna del 1954, nel quale FERMI riportò i risultati delle sue esperienze sulla diffusione dei mesoni.

La teoria nel frattempo si era anch'essa sviluppata. Dato che si conoscevano tutte le proprietà di simmetria dell'interazione fra mesoni e nucleoni, ci si attendeva che un'unica equazione di campo fornisse la spiegazione di tutti i fatti sperimentali. Non si sapeva se tale equazione ammettesse soluzioni, nè si sapeva quanto buone potessero essere le varie approssimazioni fatte assumendo l'esistenza di soluzioni. Dopo un breve periodo di ottimismo causato dall'accordo raggiunto nella spiegazione mesonica delle forze nucleari, e dopo la dimostrazione data da KLEIN che questo accordo era più che altro fortuito, si ricadde, riguardo al sistema delle equazioni mesoniche, in una specie di agnosticismo che per alcuni dura tuttora.

Il terzo periodo dello sviluppo della Fisica dei mesoni iniziò con i lavori di CHEW sulla cosiddetta teoria di sorgente fissa, nella quale si tratta l'interazione mesone-nucleone trascurando i gradi di libertà cinematici dei nucleoni. Questa teoria dava un ottimo accordo semiquantitativo (con deviazioni dell'ordine del 15%) di tutta una classe di fenomeni mesonici: la presenza della risonanza, e alcuni aspetti della fotoproduzione. Si poteva dire in questo periodo (siamo ormai nel 1955) che, data la radicale schematizzazione del modello, l'accordo era troppo buono per non essere dovuto a cause più generali.

Esse non tardarono a manifestarsi. Già nel 1954 GOLDBERGER e THIRRING avevano cercato di applicare le classiche regole di dispersione di Kramers (che furono in parte responsabili della rivoluzione della teoria quantistica nel lontano 1925) fra l'altro al problema della fotoproduzione dei mesoni. Dal quasi ovvio postulato che un segnale non può mai arrivare prima di essere partito (postulato della causalità) essi derivarono una relazione che permetteva di collegare la dipendenza dall'energia della sezione d'urto della diffusione di fotoni da parte di nucleoni con la sezione d'urto della fotoproduzione. Una previsione ricavabile da ciò (in seguito brillantemente verificata da BERNARDINI) fu che la sezione d'urto fotone-nucleone dovrebbe « preannunciare » l'esistenza dei mesoni già ad energie di per sè non sufficienti all'effettiva produzione di queste particelle. Le regole di dispersione furono poi generalizzate anche per il problema dell'urto fra due particelle dotate entrambe di massa. L'applicazione di queste idee al modello di Chew verificava quasi completamente i risultati ottenuti da CHEW. Uno degli scopi principali di questo nostro Corso è appunto di studiare le proprietà generali che sono conseguenza dell'applicazione del principio di causalità alla Fisica dei mesoni π.

Dallo studio delle regole di dispersione fu rilevato un altro aspetto importante di quest'ultimo periodo: i vari campi della Fisica non sono separabili. Mentre la filosofia suggerita dalla matrice S di Heisenberg suddivide la natura in compartimenti stagni, ognuno caratterizzato da un insieme di costanti di moto, la teoria che nasce dal principio di causalità fissa tra questi un forte

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legame. Come abbiamo già accennato, la Fisica dei pioni si preannuncia nella diffusione tra fotoni e nucleoni, in modo tale da far pensare che la Fisica delle particelle strane dovrebbe anch'essa « preannunciarsi » studiando i pioni.

In questo Corso vogliamo dunque anche vedere qual'è la parte che il pione svolge nel quadro di tutte le particelle « strane » che oggi possono prodursi copiosamente con le grandi macchine acceleratrici. Questo tentativo di integrazione ci condurrà anche a studiare non soltanto le cosiddette interazioni forti (come quella tra pioni e nucleoni), ma anche le cosiddette interazioni deboli, responsabili del decadimento delle particelle.

Con questo processo di realizzazione graduale di un quadro sempre più complesso delle proprietà dei mesoni π in seno alla famiglia di tutte le particelle conosciute, possiamo ritenere di essere ormai alla soglia di un quarto periodo dello sviluppo della Fisica dei pioni, nel quale questa espressione perderebbe praticamente il significato autonomo che ha tuttora e rappresenterebbe invece un aspetto particolare di una descrizione più generale della natura; in modo che il prossimo convegno su questo argomento possa essere chiamato ancora (come nel '54), e a ragione, « Convegno sulle particelle elementari ».

* * *

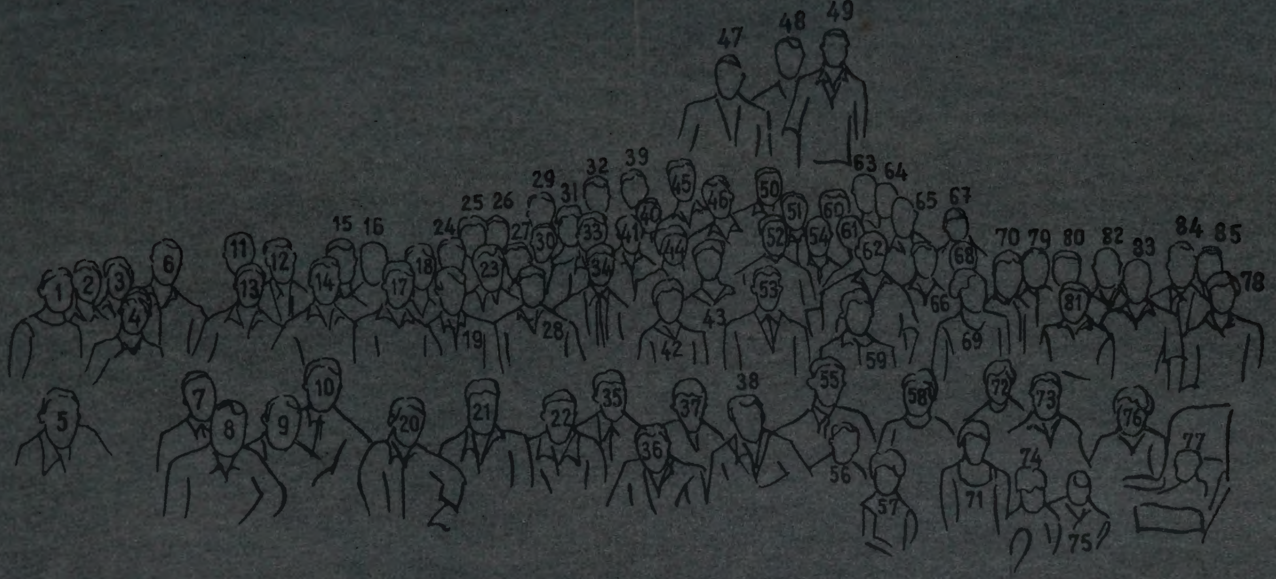
I want to state quite briefly what I think is the scope of the Course on pion physics and what we are trying to achieve.

It appears to me that we have arrived at a junction where it is no longer possible to regard pion phenomena as an isolated branch of Physics. On the other hand one can state looking backwards that that was essentially the working hypothesis to which we owe most of the progress in this field.

We propose to see more clearly in this course, where the pions stands in the general framework of elementary particles. To this end we want to give a clear picture, of what has been achieved by isolating pion-nucleon phenomena from the rest of Physics and we hope to complete this in the first week of the course.

We shall try to dedicate all the second week to the borderline (or rather the broad border strip) between Pion physics and the Physics of strange particles.

It is clear that in the brief interval we have at our disposition (about ten working days) it will not be possible to explore the whole field with lexicographic care. We rather want to pick out some of what appears to us the most important aspects of this branch of Physics and treat them as completely as possible.



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2. Sig.ra Sherr
3. Sig.na L. Sherr
4. Brisson Fouché
5. Prof. Ashkin
6. Prof. Sherr
7. Prof. Puppi
8. Ungaonkar
9. Kortez
10. Zichichi
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82. Massam
83. Prof. Thirring
84. Paul
85. Leveque

9° CORSO ESTIVO - VARENNA SUL LAGO DI COMO - VILLA MONASTERO - 18-30 Agosto 1958

LEZIONI E SEMINARI

Pion-Nucleon Scattering.

J. Ashkin

Carnegie Institute of Technology - Pittsburgh, Penna.

1. – Introduction.

The interaction of pions with nucleons forms only a certain part of the complex structure of strong interactions. Thus together with scattering processes

$$\pi^- + p \to \pi^- + p\,,$$

we also give, at sufficiently high energy

$$\begin{aligned}\pi^- + p &\to K^0 + \Lambda^0\,,\\ &\to K^+ + \Sigma^-\,,\end{aligned}$$

and other processes with appreciable cross section. It may very well turn out that a complete understanding of even the restricted phenomena involved in the scattering requires an appreciation of the entire family of strongly interacting particles. Nevertheless from an experimental point of view there is a clear separation between the low energy phenomena like pion scattering, photoproduction, production in nucleon-nucleon collisions, which we call now classical pion physics, and the high energy phenomena involving the strange particles. Also theoretically considerable progress has been made by adhering to such a separation so that there is some hope that we can at least partially digest our limited experimental knowledge at the low energy, before being forced to swallow the much larger and richer class of phenomena now being explored with the very high energy accelerators.

In the first week of the course we are therefore going to review classical pion physics. Our part will be to discuss the scattering of pions by nucleons, mainly from an experimental point of view.

Already in the presentation of the facts it will be useful to recognize the great simplification arising from the assumption of charge independence. We shall examine later the supporting evidence.

For the present purpose let us recall that according to this hypothesis neutron and proton are regarded as different manifestations (with respect to charge) of a single generic particle — the nucleon, while the particles π^+, π^- and π^0 form the three charge states available to the pion. From this point of view we may introduce a two component wave function for the nucleon with components referring to the neutron and proton states together with a three component wave function for the pion. The charge independence assumption requires that under a general unitary transformation among the different charge components of the total wave function for a system of nucleons and pions, the physical properties of the system shall remain unchanged. It is well known [1] that this invariance principle leads to a conservation law for the generators of the unitary transformations: the three components of a vector operator $\boldsymbol{I}$ in an abstract « charge space ». The total operator $\boldsymbol{I}$ is the sum of an operator $\boldsymbol{t}$ for the nucleons and an operator $\boldsymbol{T}$ for the pions, each of which has the mathematical, properties associated with an ordinary angular momentum.

We therefore call

$$\boldsymbol{I} = \boldsymbol{t} + \boldsymbol{T},$$

the total isotopic spin operator for the system.

The isotopic spin assignment for the pion and nucleon separately are in Table I.

TABLE I.

$T = 1$	$t = \frac{1}{2}$
$T_3 = +1$, π^+	$\tau_3 = \frac{1}{2}$, p
$T_3 = 0$, π^0	$\tau_3 = -\frac{1}{2}$, n
$T_3 = -1$, π^-	

For the system: nucleon+pion the total isotopic spin I evidently has the two possible values $I = \frac{3}{2}$ and $I = \frac{1}{2}$. In Table II we list for reference the isotopic spin wave function for the two cases as one obtains them from the addition rules of angular momenta [2].

TABLE II.

	$I = \frac{3}{2}$	$I = \frac{1}{2}$
$I_3 = +\frac{3}{2}$	$\lvert\pi^+ \mathrm{p}\rangle$	
$I_3 = +\frac{1}{2}$	$\sqrt{\frac{2}{3}}\lvert\pi^0 \mathrm{p}\rangle - \sqrt{\frac{1}{3}}\lvert\pi^+ \mathrm{n}\rangle$	$\sqrt{\frac{1}{3}}\lvert\pi^0 \mathrm{p}\rangle + \sqrt{\frac{2}{3}}\lvert\pi^+ \mathrm{n}\rangle$
$I_3 = -\frac{1}{2}$	$\sqrt{\frac{1}{3}}\lvert\pi^- \mathrm{p}\rangle + \sqrt{\frac{2}{3}}\lvert\pi^0 \mathrm{n}\rangle$	$\sqrt{\frac{2}{3}}\lvert\pi^- \mathrm{p}\rangle - \sqrt{\frac{1}{3}}\lvert\pi^0 \mathrm{n}\rangle$
$I_3 = -\frac{3}{2}$	$\lvert\pi^- \mathrm{n}\rangle$	

2. – Consequences of the charge independence assumption for π-p scattering.

2·1. *Total cross section.* – According to the hypothesis of charge independence, there should be some relation between the total cross section for the collision processes

$$\pi^+ + \mathrm{p} \qquad \text{and} \qquad \pi^- + \mathrm{p}\,.$$

The resolution into eigenstates of I, I_3 follows:

$$|\pi^+ + \mathrm{p}\rangle = |\tfrac{3}{2}, \tfrac{3}{2}\rangle\,,$$

$$|\pi^- + \mathrm{p}\rangle = \sqrt{\tfrac{1}{3}}\,|\tfrac{3}{2}, -\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2}, -\tfrac{1}{2}\;,$$

$|\pi^+\mathrm{p}\rangle$ is thus a pure $I = \frac{3}{2}$ state. We may therefore write

$$\sigma_{\text{tot}}(\pi^+ + \mathrm{p}) = \sigma_{\text{tot}(\frac{3}{2})} \equiv \sigma_{(\frac{3}{2})}\,,$$

$|\pi^-\mathrm{p}\rangle$ is a mixture of $I = \frac{3}{2}$ and $I = \frac{1}{2}$ with weights shown. The final state, after interaction, will likewise be a mixture of $I = \frac{3}{2}$ and $I = \frac{1}{2}$. In measuring a total cross section we *sum* over the different charge possibilities in the final state: for example

$$\begin{aligned} \pi^- + \mathrm{p} &\to \pi^- + \mathrm{p}\,,\\ &\to \pi^0 + \mathrm{n}\,,\\ &\to \pi^- + \pi^0 + \mathrm{p}\,,\\ &\to \pi^0 + \pi^0 + \mathrm{n} \quad \text{etc.}\,. \end{aligned}$$

Since the $I = \frac{3}{2}$ and $I = \frac{1}{2}$ states are orthogonal, there will be no net interference after the squares of the different reaction amplitudes are all added together. Furthermore, by hypothesis, the amplitudes cannot depend upon I_3, but only upon I, with the $I = \frac{3}{2}$ amplitude appearing with the factor $1/\sqrt{3}$ and the $I = \frac{1}{2}$ with the factor $\sqrt{\frac{2}{3}}$. It follows that

$$\sigma_{\text{tot}}(\pi^- + \mathrm{p}) = \tfrac{1}{3}\,\sigma_{(\frac{3}{2})} + \tfrac{2}{3}\,\sigma_{(\frac{1}{2})}\,.$$

The π^- total cross section should therefore exhibit the features of the π^+ total cross section, scaled down by a factor of three, and in addition should contain the features of the $I = \frac{1}{2}$ cross section. The measured total cross sections are shown in Fig. 1 [3].

It is very striking that the strong peak in the total π^+ cross section, maximum at approximately 180 MeV pion kinetic energy in the laboratory system, is in fact reproduced in the π^- cross section, reduced by a factor 3, within the accuracy. Assuming the charge independence analysis to be valid, we may conclude that in the energy region from 0 to 300 MeV the $\sigma_{(\frac{1}{2})}$ is quite small compared to $\sigma_{(\frac{3}{2})}$.

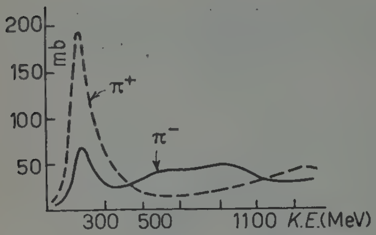

Fig. 1.

Most of our detailed quantitative information comes from this energy range. Likewise our theoretical understanding is to a large extent limited also to this region.

At still higher energies we see evidence of a broad maximum in the $I=\frac{1}{2}$ cross section at $\sim$ 900 MeV, and another maximum in the $I=\frac{3}{2}$ cross section at $\sim$ 1400 MeV. What these signify is entirely unclear. There is in addition some evidence for a strong or resonant interaction in the $I=\frac{1}{2}$ state at an energy somewhat lower than 900 MeV. This comes from recent measurements at Cornell and Cal Tech on photo-meson production with photons in the region $(500 \div 1000)$ MeV.

2·2. *Differential cross sections.* – Let us confine our attention to *elastic processes* in which there are only one pion and the nucleon in the final state. Production of an additional pion is possible for incident pion energy above 170 MeV but remains small below 250 MeV. Experimental investigation of these « inelastic » processes might prove to be of considerable interest.

The possible elastic processes are:

$$\begin{aligned}
\pi^+ + p &\to \pi^+ + p\,,\\
\pi^- + p &\to \pi^- + p\,,\\
&\to \pi^0 + n\,,\\
&\to 2\gamma\,.
\end{aligned}$$

Neglecting the mass differences between neutron and proton, and between charged and neutral pions, the amplitudes for these reactions are related by charge independence.

Since the proton has a spin, it is convenient to represent the scattering amplitude in terms of a matrix $\mathcal{M}$ which operates on the spinor for the incident wave to produce the spinor for the wave scattered in any particular direction. In the following discussion we shall treat the nucleon in a non-relativistic approximation. Only a slight modification will be necessary to extend the argument to the relativistic case.

Consider the positive pion scattering which has available only a single

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final charge state. In the center of mass system, let a proton in the spin state χ and momentum k (computed from the relativistic formulas in transforming from laboratory to center of mass system) be approaching a pion moving in the opposite direction. Choosing the z-axis as the direction of the incident proton, we may represent the incident wave function for the relative motion by

$$\psi_{\text{inc}} = \exp[ikz]\chi|\pi^+\text{p}\rangle\,.$$

Since the scattered wave in some direction θ, φ must depend linearly on χ, we may define a « scattering » matrix $\mathcal{M}(\theta, \varphi)$ by the equation

$$\psi_{\text{scatt}} \to \frac{1}{r}\exp[ikr]\mathcal{M}(\theta, \varphi)\chi|\pi^+\text{P}\rangle\,,$$

where $\mathcal{M}$ here is a 2×2 matrix operating on two component spinors χ. It is the eventual aim of the theory to predict the exact behaviour of the matrix $\mathcal{M}$. General invariance arguments suffice, however, to restrict the *form* of $\mathcal{M}$. Thus $\mathcal{M}$ must be some function of the incident proton momentum $\boldsymbol{p}_1$, final proton momentum $\boldsymbol{p}_2$, and the Pauli spin operator $\boldsymbol{\sigma}$ which is invariant under spatial rotations and reflections. It must therefore be possible to write $\mathcal{M}$ as

$$\mathcal{M}(\theta, \varphi) = f(\theta) + ig(\theta)\boldsymbol{\sigma}\cdot\boldsymbol{n}\,,$$

where θ is the angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and $\boldsymbol{n}$ is some *axial* vector required to make $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ invariant under reflections. The only axial vector which can be constructed from $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ is proportional to their cross product. We may therefore take

$$\boldsymbol{n} = (\boldsymbol{p}_2\times\boldsymbol{p}_1)/|\boldsymbol{p}_2\times\boldsymbol{p}_1|\,.$$

Notice that $\boldsymbol{n}$ is a unit vector perpendicular to the plane of the scattering. If the direction of incident momentum is chosen as the axis of quantization for the proton spin (z-axis) we see that $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ involves only σ_x and σ_y and therefore produces a flipping of the spin. For this reason we shall speak of

$g(\theta)$ as the *spin-flip amplitude*,

and

$f(\theta)$ as the *no spin-flip amplitude*.

In $\pi^+ + \text{p}$ scattering we therefore have *two* characteristic amplitudes which we shall denote by

$$f_+(\theta) = f_{\frac{3}{2}}(\theta)\,, \qquad g_+(\theta) = g_{\frac{3}{2}}(\theta)\,,$$

where the subscript $\frac{3}{2}$ reminds us that the scattering is associated with a unique state of isotopic spin $\frac{3}{2}$.

In the case of negative pion scattering *two* charge channels are available in the final state. These may be expressed in terms of eigenstates of I and I_3 by

$$|\pi^- \mathrm{p}\rangle = 1/\sqrt{3}\,|\tfrac{3}{2}, -\tfrac{1}{2}\rangle + \sqrt{2/3}\,|\tfrac{1}{2}, -\tfrac{1}{2}\rangle\,,$$

$$|\pi^0 \mathrm{n}\rangle = \sqrt{2/3}\,|\tfrac{3}{2}, -\tfrac{1}{2}\rangle - 1/\sqrt{3}\,|\tfrac{1}{2}, -\tfrac{1}{2}\rangle\,.$$

Writing $\mathcal{M}_{(\frac{3}{2})}$ and $\mathcal{M}_{(\frac{1}{2})}$ for the scattering matrices in the respective isotopic spin states, the relation between incident and scattered waves becomes

$$\exp[ikz]\,\chi\,|\pi^- \mathrm{p}\rangle \to (1/r)\exp[ikr]\{1/\sqrt{3}\,|\tfrac{3}{2}, -\tfrac{1}{2}\rangle\,\mathcal{M}_{(\frac{3}{2})} + \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2}, -\tfrac{1}{2}\rangle\mathcal{M}_{(\frac{1}{2})}\}\chi\,.$$

Expressed in terms of the charge states $|\pi^- \mathrm{p}\rangle$ and $|\pi^0 \mathrm{n}\rangle$ the scattered wave is

$$(1/r)\exp[ikr]\,|\pi^- \mathrm{p}\rangle\{\tfrac{1}{3}\,\mathcal{M}_{(\frac{3}{2})} + \tfrac{2}{3}\mathcal{M}_{(\frac{1}{2})}\}\chi + (1/r)\exp[ikr]\,|\pi^0 \mathrm{n}\rangle\{\sqrt{2}/3\,(\mathcal{M}_{(\frac{3}{2})} - \mathcal{M}_{(\frac{1}{2})})\}\chi\,.$$

We may conveniently summarize these results in Table III, where an obvious change of notation has been made.

TABLE III. – *Scattering amplitudes for pion-proton scattering.*

Process	Amplitude
$\pi^+ + \mathrm{p} \to \pi^+ + \mathrm{p}$	$\mathcal{M}_+ = \mathcal{M}_3$
$\pi^+ + \mathrm{p} \to \pi^+ + \mathrm{p}$	$\mathcal{M}_- = \frac{1}{3}\,\mathcal{M}_3 + \frac{2}{3}\,\mathcal{M}_1$
$\pi^+ + \mathrm{p} \to \pi^0 + \mathrm{n}$	$\mathcal{M}_0 = \sqrt{2}/3\,(\mathcal{M}_3 - \mathcal{M}_1)$

Each matrix $\mathcal{M}$ is expressible in terms of associated amplitudes $f(\theta)$ and $g(\theta)$.

To obtain the differential cross section we must find the norm of the scattered spinor $\mathcal{M}\chi$:

$$\chi^+\mathcal{M}^+\mathcal{M}\chi = \chi^+(f^*(\theta) - ig^*(\theta)\boldsymbol{\sigma}\cdot\boldsymbol{n})\,(f(\theta) + ig(\theta)\boldsymbol{\sigma}\cdot\boldsymbol{n})\,\chi =$$

$$= |f(\theta)|^2 + |g(\theta)|^2 + i(f^*(\theta)g(\theta) - f(\theta)g^*(\theta))\,\chi^+\boldsymbol{\sigma}\cdot\boldsymbol{n}\chi\,,$$

where we have used the fact that $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ is an Hermitian operator whose square is unity. In all experiments performed up to now the hydrogen target is unpolarized. In this case the expectation value of the proton spin, appearing in the last term, may be set equal to zero, giving for the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta)|^2 + |g(\theta)|^2. \qquad \text{(abbreviated as } |\mathcal{M}|^2\text{)}.$$

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Experimental measurements of the differential cross sections, in many cases of all three reactions, exist in the energy range from 25 MeV to 300 MeV. It is interesting to compare the experiments with the consequences to be derived from our table of scattering amplitudes.

For example, it is evident from the table that upon adding the differential cross section for elastic π^- scattering and for charge exchange scattering, the interference between the two isotopic spin amplitudes disappears, leaving

$$\frac{d\sigma_{(-)}}{d\Omega}+\frac{d\sigma_{(0)}}{d\Omega}=\frac{1}{3}|\mathfrak{M}_3|^2+\frac{2}{3}|\mathfrak{M}_1|^2=\frac{1}{3}\frac{d\sigma_{(+)}}{d\Omega}+\frac{2}{3}\frac{d\sigma_{(\frac{1}{2})}}{d\Omega}.$$

This agrees with our previous relation for the total cross sections, obtained by integrating over solid angle. Since the total cross section $\sigma_{(\frac{1}{2})}$ is small, we should expect the sum of the π^- differential cross sections to reproduce the π^+ differential cross section reduced by a factor three. A comparison of this type is made for the measurements at 150, 170 and 220 MeV. Fig. 2 shows the differential cross sections at 150 MeV [4]. The solid curves represent the cross sections obtained from a phase shift analysis. The dashed one represents three times the differential cross section for elastic scattering and charge exchange scattering of negative pions. The result is not trivial since the interference of the two isotopic spin amplitudes is not negligible for the individual π^- cross sections. In addition the measurement of the cross section $d\sigma_{(0)}/d\Omega$, giving the angular distribution of the π^0 in charge exchange scattering, is obtained only indirectly from measurements of the resulting γ-rays, $d\sigma_{(\gamma)}/d\Omega$, and involves the estimated efficiency for γ-ray detection.

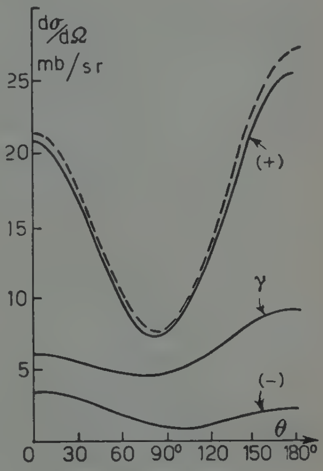
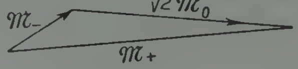


Fig. 2.

Salzman [5] has recently applied another test for examining the consistency of the data with charge independence. From the table of reaction amplitudes it follows immediately that

$$\mathfrak{M}_+=\mathfrak{M}_-+\sqrt{2}\,\mathfrak{M}_0 .$$

From the relation of $\mathfrak{M}$ to the differential cross section we see that we may regard $\mathfrak{M}$ as a vector in a space of four dimensions having the components Re $f(\theta)$, Im $f(\theta)$, Re $g(\theta)$, Im $g(\theta)$. The algebraic relation above may therefore be visualized as a geometrical relation between vectors, indicated in the diagram:



As a result the amplitudes must obey the triangle inequalities for the length of one side as compared with the sum of the remaining two. For example:

$$|\mathcal{M}_-| + |\sqrt{2}\mathcal{M}_0| \geqslant |\mathcal{M}_+|,$$

or

$$\sqrt{\frac{\mathrm{d}\sigma_{(-)}}{\mathrm{d}\Omega}} + \sqrt{2\,\frac{\mathrm{d}\sigma_{(0)}}{\mathrm{d}\Omega}} \geqslant \sqrt{\frac{\mathrm{d}\sigma_{(+)}}{\mathrm{d}\Omega}},$$

together with similar relations obtained by permuting the sides. The inequality written above is expected to be the most sensitive one since it becomes an equality in the case that the amplitude $\mathcal{M}_1$ vanishes exactly. Since $\mathcal{M}_1$ is small, the left hand side should be only slightly larger than the right hand side. For the Carnegie Tech. experiments at 150, 170 and 220 MeV there are some angles, especially at 170 MeV, for which the left hand side is *less* than the right, but not appreciably outside the experimental errors. However it should be realized that some small deviations from charge independence are to be expected in view of the mass difference between neutron and proton, and between neutral and charged pions; likewise the strength of interaction of neutral and charged pions with nucleons may very well be slightly different. Interesting attempts to discuss this problem have recently been made by GREENBERGER and SORENSEN [6] within the framework of the static model of Chew and Low [11].

In general we may say that the bulk of the experimental evidence tends to favor the hypothesis of charge independence for the pion-nucleon interaction.

3. – Phase shift analysis of pion-nucleon scattering.

As in any conventional problem of collision theory it is possible to express the cross sections for the elastic processes involving pions incident on nucleons in terms of real phase shifts characterizing partial waves, eigenstates of the constants of motion. Such a resolution into partial waves has a practical value only if the number of important partial waves remains relatively small; otherwise one may just as well remember the primitive experimental information. In the pion-proton scattering below 300 MeV it turns out that partial waves of orbital angular momentum $l = 0$ and $l = 1$ (s and p waves) are by far the most important.

This is in accord with the estimate we would make from the reasonable assumption that the range of pion-nucleon interaction is approximately the Compton wave length of the pion:

$$r_0 = \frac{\hbar}{\mu c}\,.$$

If $\lambda\!\!\!^{-}$ is the de Broglie wave length of the pion in the center of mass system we may expect that orbital angular momenta l greater than

$$l_{\max} = \frac{r_0}{\lambda\!\!\!^{-}}\,,$$

will have only minor significance for the scattering. An elementary calculation for $l_{\max}$ as a function of the pion energy in the laboratory system shows that up to 300 MeV a partial wave analysis with $l = 0$ and $l = 1$ as dominant, is likely to prove fruitful (assuming the order of magnitude of the range of interaction to be correct).

There is also some reason to expect that the p states will be of special significance. This follows from the pseudoscalar character of the pion relative to the nucleon (shown by the experiments on absorption of slow negative pions in deuterium) together with the large mass of the nucleon relative to the pion:

$$\mu/M \approx \tfrac{1}{7}\,.$$

To the extent that one can regard the nucleon as having an infinite mass, and therefore at rest with angular momentum $\frac{1}{2}$, we see that it is possible for the nucleon to emit and absorb only p-wave pions if angular momentum and parity are to be conserved. In fact, as the data show, the striking peak in the π^+ total cross section near 200 MeV, and the accompanying peak in the π^- cross section, are well explained by the hypothesis of a resonance in the p-state of total angular momentum $J = \frac{3}{2}$ and isotopic spin $\frac{3}{2}$.

Let us briefly consider the derivation of the expressions for the scattering amplitude in terms of phase shifts; that is, we wish to find expressions for the functions $f(\theta)$ and $g(\theta)$ which are associated with the matrix $\mathcal{M}(\theta, \varphi)$. We treat the proton non-relativistically. To have a state of definite total isotopic spin, it will be simpler to begin with the scattering of positive pions. More precisely suppose we have a proton of momentum k incident in the z-direction in the center of mass system, a pion incident in the negative z-direction, with the proton spin pointing along the z-axis. The incident wave function representing the relative motion is then

$$\psi_{\text{inc}} = \exp[ikz]\begin{vmatrix} 1 \\ 0 \end{vmatrix}\,.$$

Our first problem is to write this plane wave as a linear combination of free particle wave functions which are eigenstates of the quantities which are conserved in the pion-proton interaction. As constants of the motion, mutually commuting with one another, we take J^2, J_z, and the *parity* operation, where J refers to the total angular momentum. There is ample evidence for the conservation of parity to a very high degree in the strong interactions and we shall assume that reflection symmetry is obeyed exactly. For a given value of J (total angular momentum) we have associated two possible values of l (orbital angular momentum): $l = J - \frac{1}{2}$ and $l = J + \frac{1}{2}$ with opposite parities. Replacing the spherical harmonics $Y_{\rm lm}$ appropriate for particles of zero spin, we have the simultaneous eigenstates of J^2, L^2, and J_z, the functions $F^m_{l,j}$ defined by

$$F^{\frac{1}{2}}_{l,l+\frac{1}{2}} = \frac{1}{\sqrt{2l+1}} \left| \begin{matrix} \sqrt{l+1} & Y_{l,0}(\theta,\varphi) \\ \sqrt{l} & Y_{l,1}(\theta,\varphi) \end{matrix} \right| \quad \text{and} \quad F^{\frac{1}{2}}_{l,l-\frac{1}{2}} = \frac{1}{\sqrt{2l+1}} \left| \begin{matrix} \sqrt{l} & Y_{l,0}(\theta,\varphi) \\ -\sqrt{l+1} & Y_{l,1}(\theta,\varphi) \end{matrix} \right|,$$

for the special case $J_z = m = \frac{1}{2}$. Alternatively

$$Y_{l,0}(\theta,\varphi) \left| \begin{matrix} 1 \\ 0 \end{matrix} \right| = \sqrt{\frac{l+1}{2l+1}}\, F^{\frac{1}{2}}_{l,l+\frac{1}{2}} + \sqrt{\frac{l}{2l+1}}\, F^{\frac{1}{2}}_{l,l-\frac{1}{2}} .$$

The well-known expansion of $\exp[ikz]$ in spherical harmonics and spherical Bessel functions $j_l(kr)$ therefore becomes transformed into:

$$\exp[ikz] \left| \begin{matrix} 1 \\ 0 \end{matrix} \right| = \sqrt{4\pi} \sum_{l=0}^{\infty} \sqrt{2l+1}\, i^l j_l(kr) Y_{l,0}(\theta,\varphi) \left| \begin{matrix} 1 \\ 0 \end{matrix} \right| =$$

$$= \sqrt{4\pi} \sum_{l=0}^{\infty} i^l j_l(kr) \left\{ \sqrt{l+1}\, F^{\frac{1}{2}}_{l,l+\frac{1}{2}} + \sqrt{l}\, F^{\frac{1}{2}}_{l,l-\frac{1}{2}} \right\}.$$

In the scattering problem we are interested in the asymptotic behaviour of the wave function as $r \to \infty$. From the behaviour of the function $j_l(kr)$

$$j_l(kr) \to \frac{1}{kr} \sin\left(kr - \frac{1}{2} l\pi\right),$$

we find that the *outgoing* part of the incident wave has the form

$$\left\{ \left| \begin{matrix} 1 \\ 0 \end{matrix} \right| \exp[ikz] \right\}_{\rm out} \to \frac{\exp[ikr]}{r} \frac{\sqrt{4\pi}}{2ik} \sum_{l=0}^{\infty} \left\{ \sqrt{l+1}\, F^{\frac{1}{2}}_{l,l+\frac{1}{2}} + \sqrt{l}\, F^{\frac{1}{2}}_{l,l-\frac{1}{2}} \right\}.$$

Considering now the outgoing part of the *complete* wave function, we must expect that for elastic processes the only effect of the pion-proton interaction

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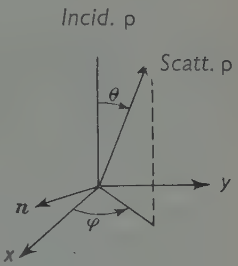
is to modify the phase of each partial wave without changing its absolute magnitude. The possibility of maintaining such an expansion in the function $F_{l,j}$ when the whole interaction is included follows simply from the assumed conservation laws. Let us denote the phase factor for the partial wave l, j conventionally by

$$\exp[2i\delta_{l,j}] .$$

To obtain the asymptotic form of the scattered wave we need only subtract from the complete wave function the asymptotic form of the incident wave, giving



$$\psi_{\text{scatt}} \rightarrow \frac{\exp[ikr]}{r}\,\frac{\sqrt{4\pi}}{2ik}\sum_{l=0}^{\infty}\left\{\sqrt{l+1}\,(\exp[2i\delta_{l,l+\frac{1}{2}}]-1)\,F^{\frac{1}{2}}_{l,l+\frac{1}{2}}+\right.$$
$$\left.+\sqrt{l}\,(\exp[2i\delta_{l,l-\frac{1}{2}}]-1)\,F^{\frac{1}{2}}_{l,l-\frac{1}{2}}\right\}.$$

This is to be compared with the result of operating on the incident spinor with the matrix $\mathcal{M}$ introduced earlier. Defining the unit normal to the plane of scattering as before

$$\boldsymbol{n} = (\sin\varphi, -\cos\varphi, 0) ,$$

we have

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\sin\varphi - \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}\cos\varphi = \begin{vmatrix} 0 & i\exp[-i\varphi] \\ -i\exp[i\varphi] & 0 \end{vmatrix} .$$

Therefore

$$\mathcal{M}\begin{vmatrix} 1 \\ 0 \end{vmatrix} = \underbrace{\begin{vmatrix} f, & -g\exp[-i\varphi] \\ g\exp[i\varphi], & f \end{vmatrix}}_{f(\theta)+ig(\theta)\boldsymbol{\sigma}\cdot\boldsymbol{n}}\begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} f(\theta) \\ g(\theta)\exp[i\varphi] \end{vmatrix} .$$

From this expression it is clear why $f(\theta)$ is called the no-spin-flip amplitude and $g(\theta)$ the spin-flip amplitude (but one should note that these designations are adapted to the case that the incident proton direction is chosen as the axis of quantization for the spin). Passage from $F_{l,j}$ back to the spherical harmonics gives finally

$$f(\theta)=\frac{\sqrt{4\pi}}{2ik}\sum_{l=0}^{\infty}\frac{1}{\sqrt{2l+1}}\left\{(l+1)\,\Delta_{l,l+\frac{1}{2}}+l\Delta_{l,l-\frac{1}{2}}\right\}Y_{l,0}(\theta)=$$
$$=\frac{1}{2ik}\sum_{l=0}^{\infty}\left\{(l+1)\Delta_{l,l+\frac{1}{2}}+l\Delta_{l,l-\frac{1}{2}}\right\}P_l(\cos\theta) ,$$

$$g(\theta)=\frac{\sqrt{4\pi}}{2ik}\sum_{l=0}^{\infty}\sqrt{\frac{l(l+1)}{2l+1}}\left\{\Delta_{l,l+\frac{1}{2}}-\Delta_{l,l-\frac{1}{2}}\right\}Y_{l,1}(\theta,\varphi)\exp[-i\varphi] ,$$

where we have used the abbreviation

$$\Delta \equiv \exp[2i\delta] - 1 = 2i \exp[i\delta] \sin\delta .$$

We may observe that in the case that $\Delta_{l,l+\frac{1}{2}} = \Delta_{l,l-\frac{1}{2}}$, which would occur if the interaction in a state of given l were independent of the proton spin orientation, these expressions reduce to the usual one for $f(\theta)$ and give zero for $g(\theta)$, as should be expected.

For later use we may note here a relation between the total cross section and the amplitude for scattering in the forward direction ($\theta = 0$). General arguments based on the conservation of probability show that the total cross section must be proportional to the imaginary part of the amplitude for coherent elastic scattering in the forward direction. (The decrease in amplitude of the transmitted wave in virtue of the interaction with the scattering center must come from an interference of the incident wave with the elastically scattered wave in the forward direction). In our case the coherent scattering amplitude is $f(\theta)$, and the « *optical theorem* », as it is called becomes

$$\sigma_{\text{total}} = \frac{4\pi}{k} \operatorname{Im} f(0) .$$

By integration of the differential cross section and calculation of $\operatorname{Im} f(0)$ we may check the theorem directly. We find

$$\sigma_{\text{total}} = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} \left\{ (l+1) \sin^2 \delta_{l,l+\frac{1}{2}} + l \sin^2 \delta_{l,l-\frac{1}{2}} \right\} .$$

This expression for the total cross section in terms of the phase shifts is no longer valid when inelastic processes become important (complex phase shifts) but the optical theorem remains correct.

For the π^+ scattering we have a set of phase shifts $\delta_{l,j}$ characteristic of isotopic spin $\frac{3}{2}$. There is a different set associated with isotopic spin $\frac{1}{2}$, with similar expressions for the amplitude $f_{\frac{1}{2}}(\theta)$ and $g_{\frac{1}{2}}(\theta)$. In accordance with our table, the amplitudes for π^- elastic and charge exchange scattering are

$$\begin{aligned} f_-(\theta) &= \tfrac{1}{3} f_{\frac{3}{2}}(\theta) + \tfrac{2}{3} f_{\frac{1}{2}}(\theta) , \\ g_-(\theta) &= \tfrac{1}{3} g_{\frac{3}{2}}(\theta) + \tfrac{2}{3} g_{\frac{1}{2}}(\theta) , \\ f_0(\theta) &= \sqrt{2}/3 \left(f_{\frac{3}{2}}(\theta) - f_{\frac{1}{2}}(\theta) \right) , \\ g_0(\theta) &= \sqrt{2}/3 \left(g_{\frac{3}{2}}(\theta) - g_{\frac{1}{2}}(\theta) \right) . \end{aligned}$$

In the discussion so far we have omitted the complication due to the Coulomb interaction between charged pion and proton. In a certain approximation we may take into account the Coulomb force by adding the Coulomb scattering amplitude to the « nuclear » scattering amplitude treated above. Since the principal amplitude for Coulomb scattering is associated with no flipping of the spin, the experimental differential cross section, for π^+ scattering as an example, is represented by

$$\frac{\mathrm{d}\sigma_{(+)}}{\mathrm{d}\Omega} = |f_+(\theta) + f_{\mathrm{Coul}}(\theta)|^2 + |g_+(\theta)|^2 .$$

At small angles the dominant contribution is from the pure Coulomb scattering represented by the square of the Coulomb amplitude. At intermediate angles, up to 30° or 40° at 150 MeV, and for larger angles at lower energies, the interference between the Coulomb and nuclear amplitudes is important. An important application of the interference is the determination of the *sign* of the nuclear amplitude (or the sign of the scattering phase shifts). Thus, from the destructive character of the interference effect near 100 MeV, as shown for example by the work of the Bologna group we may conclude that the nuclear amplitude corresponds to an attraction at this energy.

Finally we should remark on the modification of the phase shift analysis arising from the use of the Dirac equation for the proton instead of the non-relativistic Pauli equation. As it actually turns out, the expression for the differential cross section as a sum of squares of amplitudes $f(\theta)$ and $g(\theta)$ together with the partial wave expansion of these amplitudes requires no modification at all. This depends upon a simple relation between the large and small components of the solution of the Dirac equation for a free particle. The Dirac equation for the four component spinor wave function ψ, corresponding to total energy E, has the form

$$(\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta M)\psi = E\psi , \qquad \boldsymbol{p} = -i\boldsymbol{\nabla} .$$

We may write ψ explicitly as a column vector made of two 2-componet spinors ψ_a and ψ_b, the « large » and « small » components. Combining with the usual representations of the Dirac matrices

$$\boldsymbol{\alpha} = \begin{vmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{vmatrix}, \qquad \beta = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix},$$

where $\boldsymbol{\sigma}$ are the 2×2 Pauli spin matrices and 1 is the unit 2×2 matrix, the Dirac equation may be written

$$\left\{ \begin{vmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{vmatrix} \cdot \boldsymbol{p} + M \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \right\} \begin{vmatrix} \psi_a \\ \psi_b \end{vmatrix} = E \begin{vmatrix} \psi_a \\ \psi_b \end{vmatrix} .$$

Thus

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\psi_b + M\psi_a = E\psi_a\,,$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\psi_a - M\psi_b = E\psi_b\,.$$

Consequently:

$$\psi_b = \frac{1}{E+M}\,\boldsymbol{\sigma}\cdot\boldsymbol{p}\,\psi_a\,.$$

Now the scattering process is a transition from an initial state of momentum $\boldsymbol{p}_1$, to a final state of momentum $\boldsymbol{p}_2$:

$$\psi_{\text{inc}} = \left|\begin{matrix}\psi_{a,\,\text{inc}}\\[1ex] \dfrac{1}{E+M}\,\boldsymbol{\sigma}\cdot\boldsymbol{p}_1\psi_{a,\,\text{inc}}\end{matrix}\right|, \qquad \psi_{\text{scatt}} = \left|\begin{matrix}\psi_{a,\,\text{scatt}}\\[1ex] \dfrac{1}{E+M}\,\boldsymbol{\sigma}\cdot\,\boldsymbol{p}_2\,\psi_{a,\,\text{scatt}}\end{matrix}\right|.$$

The cross section is obtained by comparing incident and scattered intensities.

$$\psi^{+}_{\text{inc}}\,\psi_{\text{inc}} = \left|\,\psi^{+}_{a,\,\text{inc}}\ \ \psi^{+}_{a,\,\text{inc}}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_1}{E+M}\right| \left|\begin{matrix}\psi_{a,\,\text{inc}}\\[1ex] \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_1}{E+M}\,\psi_{a,\,\text{inc}}\end{matrix}\right| = \psi^{+}_{a,\,\text{inc}}\psi_{a,\,\text{inc}}\left\{1+\frac{p_1^2}{(E+M)^2}\right\},$$

since $(\boldsymbol{\sigma}\cdot\boldsymbol{p}_1)^2 = p_1^2$. In the same way the scattered intensity is proportional to the square of the large component of the scattered wave, with the *same* proportionality factor ($p_2^2 = p_1^2$). For calculating the cross section it is therefore sufficient to compare incident and scattered intensities for the large components alone.

Just as for the non-relativistic case one expands the incident plane wave into partial waves, eigenstates of J^2, J_z and *parity*, and finds the scattered wave by shifting the phase of the outgoing part of each partial wave. Since the Dirac equation must reduce to the Pauli equation in the non-relativistic limit, it is not surprising that in this procedure the relation between the large components is *exactly* the same as in the non-relativistic treatment. Consequently one finds

$$\text{Incident spinor} = \left|\begin{matrix}\chi\\[1ex] \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_1}{E+M}\,\chi_{\text{inc}}\end{matrix}\right| \to \text{Scattered spinor} = \left|\begin{matrix}\mathcal{M}(\theta,\varphi)\,\chi_{\text{inc}}\\[1ex] \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_2}{E+M}\,\mathcal{M}(\theta,\varphi)\,\chi_{\text{inc}}\end{matrix}\right|,$$

where the 2×2 matrix $\mathcal{M}(\theta,\varphi)$ has the same expression as previously. In particular, in the forward direction ($\theta = 0$)

$$\mathcal{M}(\theta = 0) = f(0)\cdot 1\,,$$

since $g(\theta)$ vanishes like $\sin\theta$ for $\theta \to 0$. Therefore

$$\text{forward scattered spinor} = f(0)\cdot(\text{incident spinor})\,.$$

Thus $f(0)$ retains its significance as the forward scattering amplitude, and enters into the optical theorem in the same manner as before.

4. – Restriction to *s*-and *p*-waves.

As anticipated earlier, the scattering below 300 MeV appears to be well represented in terms of partial waves with $l=0$ and $l=1$. There is perhaps some indication of the presence of d-waves at the higher energies ~ 300 MeV.

If d-waves are neglected we are left with three angular momentum states: $s_{\frac{1}{2}}$, $p_{\frac{1}{2}}$, and $p_{\frac{3}{2}}$. With each angular momentum state we may have isotopic spin $\frac{3}{2}$ or $\frac{1}{2}$ giving a total of six states for which the phase shifts should describe the scattering. It is customary to write these phase shifts in the same notation as originally employed by Fermi in his first discussion of this problem. The notation is indicated in the following Table.

Angular momentum state	Phase shift	
	$I=\frac{3}{2}$	$I=\frac{1}{2}$
$s_{\frac{1}{2}}$: $l=0$, $j=\frac{1}{2}$	α_3	α_1
$p_{\frac{1}{2}}$: $l=1$, $j=\frac{1}{2}$	α_{31}	α_{11}
$p_{\frac{3}{2}}$: $l=1$, $j=\frac{3}{2}$	α_{33}	α_{13}

Writing

$$\Delta_3 \equiv \exp[2i\alpha_3] - 1\,,$$

$$\Delta_{31} \equiv \exp[2i\alpha_{31}] - 1\,,$$

$$\Delta_{33} \equiv \exp[2i\alpha_{33}] - 1\,,$$

the amplitudes $f(\theta)$ and $g(\theta)$ for isotopic spin $\frac{3}{2}$ become

$$f_3(\theta) = \frac{1}{2ik}\{\Delta_3 + (2\Delta_{33} + \Delta_{31})\cos\theta\}\,,$$

$$g_3(\theta) = -\frac{1}{2ik}(\Delta_{33} - \Delta_{31})\sin\theta\,.$$

For isotopic spin $\frac{1}{2}$ the expressions are the same except that Δ_{31} is replaced by Δ_{11}, Δ_{33} by Δ_{13}, and Δ_3 by Δ_1.

From the form of $f(\theta)$ and $g(\theta)$ we see that the differential cross section must have the form

$$\frac{d\sigma}{d\Omega} = a + b\cos\theta + c\cos^2\theta\,.$$

If an appreciable amount of d-wave were present we would need also terms in $\cos^3\theta$ and $\cos^4\theta$. For most of the experimental data it appears that the representation in terms of s- and p-waves is adequate. (At the low energies the Coulomb interference requires a modification of the form of the cross section in the manner described earlier.) From the differential cross section for the three reactions we obtain by least squares adjustments the best value of the coefficients a, b, c, giving a total of nine quantities.

If our assumptions are correct it should be possible to represent these nine coefficients in terms of the six phase shifts we have introduced. In this way we are provided with a test of the charge independence hypothesis, which will become more rigorous as the experimental accuracy improves. In general the experimental data are well represented by six real numbers varying in an appropriate manner over the energy range up to 300 MeV. However, if one considers the experimental errors the complex mathematical problem of finding the best set of six phase shifts does not have a unique solution. Nevertheless, it now appears that the general features of the energy variation of the more important phase shifts are given clearly by the experiments.

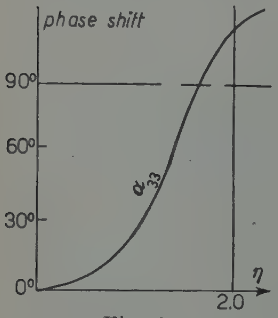


Fig. 3.

Other phase shift solutions, such as the one whose existence was first noticed by YANG [7], have been excluded theoretically by an application of the dispersion relations [8] and experimentally appear unlikely on the basis of a measurement of the polarization of the recoil protons in the pion scattering [9].

The dominant feature of the indicated experimental phase shifts is the spectacular growth of the phase shift α_{33} for the $p_{\frac{3}{2}}$ state of $I = \frac{3}{2}$, which passes steeply through 90° at approximately 190 MeV pion laboratory energy (see Fig. 3). The resonance for this state is already evident from the energy variation of the total cross sections where the value near the peak is practically all accounted for by the resonant contribution. From our previous expression for the total cross section we see that α_{33} contributes

$$\sigma_{33} = \frac{8\pi}{k^2}\sin^2\alpha_{33}\,,$$

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to the π^+ total cross section. Near the peak the experimental cross section is only slightly in excess of the resonant value $8\pi\lambda\!\!\!^{-2}$, indicating that the other phase shifts are comparatively small. Beyond the peak the drop in the cross section is associated with the growth of α_{33} above 90°. At low energies α_{33} increases in proportion to the third power of the pion momentum in the center of mass system, as should be the case for a p-wave phase shift. This is shown quite convincingly by the low energy scattering experiments at Rochester and Columbia. From the data at 41.5 MeV the Rochester group [10] finds

$$\frac{\alpha_{33}}{\eta^3} = 0.234 \pm 0.019\,,$$

where η is the pion momentum in the center of mass system in units of μc.

It is one of the most important successes of the static theory of Chew and Low [11] and of its extension using the relativistic dispersion relations [12] that the resonance behavior of α_{33} was predicted. According to the theory the phase shift α_{33} should satisfy an effective range formula (to a good approximation)

$$\frac{\eta^3}{\omega^*}\cot\alpha_{33} = \frac{3}{4f^2}\,(1 - r_{33}\omega^*)\,.$$

Here ω^* is the total energy of the (π, p) system, excluding the proton rest energy, in units of μc^2, f^2 is the (renormalized) coupling constant for the pion-nucleon interaction, and r_{33} is the effective range. The theory asserts that r_{33} is necessarily positive but is unable to fix its precise value. Positive r_{33}, however, is sufficient to insure a resonance; for, α_{33} will equal 90° for the resonance energy

$$\omega_0^* = 1/r_{33}\,.$$

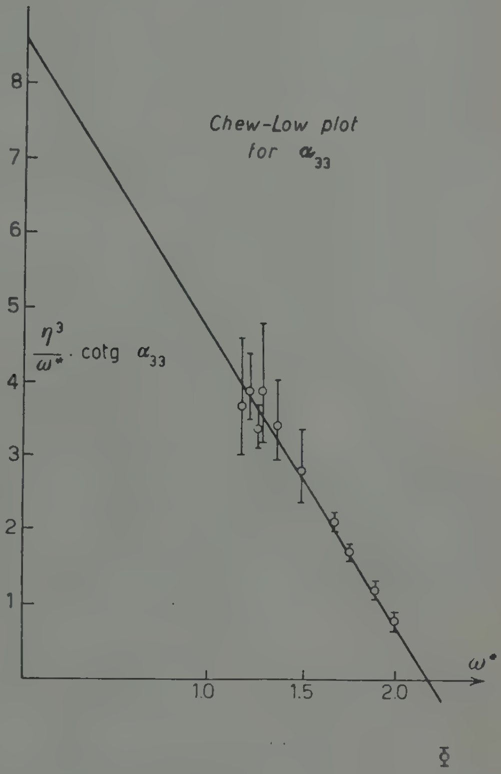

Fig. 4.

The Fig. 4 (taken from ref. [10]) shows that the experimental α_{33} does indeed satisfy a relation of the desired type. From the intercept at $\omega^* = 0$ of the

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straight line which gives a best fit with the data one finds a value for the coupling constant f^2, and from the place where it crosses the ω^* axis we get the resonance energy. One finds

$$f^2 \approx 0.08 \qquad \text{and} \qquad \omega_0^* \approx 2.2\mu c^2 .$$

The situation with regard to the remaining phase shifts is not as satisfactory but has been much improved recently by the accurate low energy experiments at Rochester [10]. The phase shift α_3 varies linearly with pion momentum over the range of measured energies from 25 MeV to 60 MeV; the best value at present is

$$\alpha_3 = (-0.110 \pm 0.004)\eta .$$

Such a linear behavior with momentum is to be expected for an s-wave if the de Broglie wave length is large compared with the range of the interaction. From the theory there is no indication of the significance of the sign of α_3 or its numerical magnitude although this must surely reveal an important feature of the pion-proton interaction. There is also an indication that the linear variation of α_3 with η is maintained at higher energies up to 300 MeV. This may indicate a range of interaction in this state considerably shorter than $\hbar/\mu c$ but the conclusion is not yet certain.

The s-wave phase shift α_1 is positive and also varies linearly in the low energy region as far as the scattering experiments can determine. An average of existing measurements gives [13]:

$$\alpha_1 = (0.173 \pm 0.011)\eta .$$

At higher energy it appears that α_1 is leveling off.

Of the three remaining p-wave phase shifts, the best determined is α_{31} From the experiments at 24.8 MeV, 31.5 MeV, and 41.5 MeV at Rochester [10] combined with those at 58 MeV and 65 MeV at Columbia [14], one finds

$$\frac{\alpha_{31}}{\eta^3} = -0.044 \pm 0.005 .$$

The result is not inconsistent with the approximate predictions of the static theory or its extension by use of the dispersion relations. These predictions are in form of effective range relations

$$\lambda_\alpha \frac{\eta^3}{\omega^*} \cot \delta_\alpha \approx 1 - \omega^* r_\alpha ,$$

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with

$$\lambda_{11} = -\tfrac{8}{3} f^2 , \qquad \lambda_{33} = \tfrac{2}{3} f^2 , \qquad \lambda_{13} = \lambda_{31} = -\tfrac{4}{3} f^2 ,$$

$$r_{33} = 1/\omega_0^* , \qquad r_{11} \approx r_{13} \approx r_{31} \approx -r_{33} .$$

Here we have written δ_α for the phase shift associated with a state α (= 33, 31, 13, 11). The negative value for λ and the negative effective range for the 31 state indicate that the phase shift α_{31} should be small, negative, and should increase as η^3. However, in obtaining these theoretical estimates, one has neglected sizable corrections coming from integrals over the very high energy cross sections which enter the dispersion relations. It is therefore not clear whether one can accept these estimates even for order of magnitude.

According to the above relations we should also have α_{13} and α_{11} negative, with α_{13} of the same order as α_{31}, and α_{11} roughly four times larger. The experiments up to 65 MeV give some indication that α_{13} and α_{11} are negative but the uncertainties are very large. At higher energies the magnitudes and signs of these phase shifts are essentially undetermined. We only know that they are small.

With regard to the determination of the small phase shifts, we can imagine an improvement coming from increased experimental accuracy. The recent measurements of π^- elastic scattering at Liverpool [15], where cross sections at 13 angles were measured to an accuracy of 4 to 5%, can be given as a good example. Phase shifts based on these measurements have not yet been published but it is important to remark that along with a better experimental accuracy there comes a demand for more theoretical precision in the analysis of the data. Small terms failing to conserve isotopic spin may well have an important influence on the small phases. From the work of GREENBERGER and SORENSEN [6], based on the static model of Chew and Low [11], it seems that a purely charge independent analysis of the experimental data below the resonance is not sufficient as far as the small p-wave or even the s-wave phase shifts are concerned.

Another experimental method which can be used for obtaining information about the small phase shifts is the measurement of the polarisation of the recoil proton in pion scattering. The proton will in general be polarized perpendicular to the plane of scattering, the magnitude of the polarization being given in terms of $f(\theta)$ and $g(\theta)$ by

$$\boldsymbol{n}\cdot P(\theta) = i \, \frac{-f(\theta)\, g^*(\theta) + f^*(\theta)\, g(\theta)}{|f(\theta)|^2 + |g(\theta)|^2} \, \boldsymbol{n} .$$

The polarization thus represents an interference between the two amplitudes $f(\theta)$ and $g(\theta)$, and is somewhat more sensitive to the small phase shifts than is

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the differential scattering cross section. As the experiments improve this may turn out to be an important addition to our information. Measurements are difficult since they involve examining a left-right asymmetry in a subsequent scattering of protons produced in a primary scattering event. The existing measurements of polarization in $\pi^- + p$ scattering at 220 MeV [9] are sufficiently accurate to discriminate between some of the grossly different phase shift solutions which fit the differential cross sections, but they are not accurate enough to give details of the small phase shifts.

APPENDIX

Deduction of the expression for the polarization.

The polarization of the recoil proton is defined as the expectation value of the Pauli spin operator $\boldsymbol{\sigma}$ for the spin states of the scattered wave. With the target protons initially unpolarized, it is clear from symmetry considerations that the polarization of the outgoing protons must be in the direction $\boldsymbol{n}$ of the normal to the plane of scattering. It will therefore be convenient to choose this direction $\boldsymbol{n}$ as axis of quantization for the spin and to regard the initial unpolarized protons in the target as being in a state which is an incoherent and equal mixture of the two (normalized) states χ_+ and χ_- for the spin parallel and antiparallel to $\boldsymbol{n}$. In accordance with the scattering matrix $\mathcal{M}(\theta,\varphi)$, which has the form $f(\theta) + ig(\theta)\boldsymbol{\sigma}\cdot\boldsymbol{n}$, the initial states χ_+ or χ_- give rise to the outgoing spinors which are given by

$$\chi_+ \to \chi_+^{sc} = (f(\theta) + ig(\theta)\,\boldsymbol{\sigma}\cdot\boldsymbol{n})\chi_+ = \big(f(\theta) + ig(\theta)\big)\chi_+ \,,$$
$$\chi_- \to \chi_-^{sc} = (f(\theta) + ig(\theta)\,\boldsymbol{\sigma}\cdot\boldsymbol{n})\chi_- = \big(f(\theta) - ig(\theta)\big)\chi_- \,.$$

The expectation value of $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ is then

$$\langle\boldsymbol{\sigma}\cdot\boldsymbol{n}\rangle = \frac{\chi_+^{sc\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{n}\chi_+^{sc} + \chi_-^{sc\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{n}\chi_-^{sc}}{\chi_+^{sc\dagger}\chi_+^{sc} + \chi_-^{sc\dagger}\chi_-^{sc}} =$$
$$= \frac{(f^* - ig^*)(f + ig) - (f^* + ig^*)(f - ig)}{(f^* - ig^*)(f + ig) + (f^* + ig^*)(f - ig)} = i\,\frac{f^*g - g^*f}{|f|^2 + |g|^2}\,.$$

Therefore

$$\text{Polarization } \langle\boldsymbol{\sigma}\rangle \equiv \boldsymbol{n}P(\theta) = i\,\frac{f^*(\theta)\,g(\theta) - f(\theta)\,g^*(\theta)}{|f(\theta)|^2 + |g(\theta)|^2}\,\boldsymbol{n}\,.$$

$P(\theta)$ is taken as positive if the polarization is taken in the direction

$$\boldsymbol{\pi}_{\text{scatt}} \times \boldsymbol{\pi}_{\text{inc}}\,,$$

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or equivalently in the direction

$$\boldsymbol{\pi}_{\text{inc}} \times \boldsymbol{p}_{\text{scatt}} ,$$

where $\boldsymbol{\pi}_{\text{inc}}$ and $\boldsymbol{\pi}_{\text{scatt}}$ refer to the momenta of the incident and scattered pions respectively and $\boldsymbol{p}_{\text{scatt}}$ refers to the momentum of the scattered proton.

It may be seen from the expressions for the scattering amplitudes in terms of phase shifts that $f(\theta)$ and $g(\theta)$ will be complex numbers of the same phase if *all* the scattering takes place in only one state of angular momentum and isotopic spin (for example the $\frac{3}{2}$, $\frac{3}{2}$ state). In this case the polarization will vanish identically. Small contributions to the scattering from other states may therefore be detected with somewhat greater sensitivity in the polarization than in the differential cross section.

REFERENCES

[1] K. M. Watson: *Phys. Rev.*, **85**, 852 (1952); E. Fermi: *Suppl. Nuovo Cimento*, **2**, 17 (1955).

[2] H. A. Bethe and F. de Hoffmann: *Mesons and Fields*, vol. **2** (Illinois), p. 60.

[3] S. J. Lindenbaum: *Ann. Rev. Nucl. Sci.*, **7**, 317 (1957), Fig. 3.1.

[4] J. Ashkin, J. P. Blaser, F. Feiner and M. O. Stern: *CERN Symposium*, **2**, 226 (1956).

[5] H. J. Schnitzer and G. Salzman: *Phys. Rev.*, **112**, 1802 (1958).

[6] J. Greenberger: quoted by C. F. Chew in *CERN Conference* (1958), p. 100; R. A. Sorensen: *Phys. Rev.*, **112**, 1813 (1958).

[7] See ref. [2], p. 72. Another type of ambiguity has been pointed out by S. Minami: *Progr. Theor. Phys. Japan*, **11**, 213 (1954).

[8] W. C. Davidon and M. L. Goldberger: *Phys. Rev.*, **104**, 1119 (1956); W. Gilbert and G. Screaton: *Phys. Rev.*, **104**, 1758 (1956); S. J. Lindenbaum and R. Sternheimer: *Phys. Rev.*, **110**, 1174 (1958).

[9] J. Ashkin, J. P. Blaser, A. Burger, J. Kunze and T. Romanowski: *CERN Conference* (1958), p. 42.

[10] S. W. Barnes, B. Rose, G. Giacomelli, J. Ring and K. Miyake: *Rochester Report*, NYO-2170 and *CERN Conference* (1958), p. 39.

[11] G. F. Chew and F. E. Low: *Phys. Rev.*, **101**, 1570 (1956).

[12] G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu: *Phys. Rev.*, **106**, 1337 (1957).

[13] G. Puppi: *CERN Conference* (1958), p. 42.

[14] D. Bodansky, A. M. Sachs and J. Steinberger: *Phys. Rev.*, **93**, 1369 (1954); **90**, 996 (1953); H. Winick: *Thesis* (Columbia University).

[15] D. N. Edwards, S. G. F. Frank and J. R. Holt: to be published.

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SUPPLEMENTO AL VOLUME XIV, SERIE X
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Fixed-Source Meson Theory.

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In the following three Lectures I want to give a brief outline of the mathematical formalism underlying the fixed source theory of the interaction between « pions » and nucleons. Various excellent accounts of this subject are found in the literature: The articles of Chew [1] to whose initiative we owe this field of research, of Chew and Low [2] (in which the effective range approximation for the description of pion-nucleon scattering is introduced) as well as the papers by Wick [3] and Low [4] in the *Reviews of Modern Physics* are readily accessible and it might be suggested to « take them as read » in a summer school on this subject. If regardless I take the risk of boring some of you out into the lake, it is because I feel that the following account is necessary for the completeness of this course in the same way as I found its preparation necessary for the purpose of steeling myself to brave the rigours of pion physics.

I will exclude from the discussion all the possible applications of the theory. These as well as the extensions to a not quite so fixed source theory, in which the recoil of the nucleon is taken into account, I will leave to others who are more qualified. Here I will only consider its basical concepts and discuss those features of the theory which one might expect to hold also in a more general formulation. I will show in particular, how dispersion relations can be derived for quantities closely related to the S-matrix. Given the shortage of time it will not be possible to develop all the physical aspects of the theory. The discussion will be restricted to apply to a small domain in the Hilbert space of the system: the domain formed from the states of one nucleon and one nucleon plus a meson. The dispersion relations derived are therefore those for pion nucleon scattering. It is admitted that this is doing injustice to a theory which has proven an equally useful tool in the discussion of nuclear structure (about which we shall hear something in a seminar to be given by

CINI), nuclear forces and photo nucleonic as well as photo mesonic interactions (which will form the subject of BERNARDINI's contribution). We shall also hear more about dispersion relations in general in the lectures which will be given by CORINALDESI and THIRRING, and PUPPI will talk about their comparison with present experimental data.

1. – Description of the nucleon.

In the simplest version of the fixed source theory it is assumed that the nucleon possesses no kinematical degrees of freedom. It appears in 4 states characterized by their spin and isotopic spin. The state vector of this « bare » nucleon χ_{st} is defined by the equations

$$\sigma_3\chi_{st} = s\chi_{st}\,, \qquad \tau_3\chi_{st} = t\chi_{st}\,, \tag{1}$$

where σ_3 is the z-component of the spin and τ_3 the charge component of the isotopic spin. s and t can assume the values ± 1 and $(s,t) = (+\,+, -\,+, +\,-, -\,-)$ correspond respectively to a proton with spin up, a proton with spin down, a neutron with spin up and a neutron with spin down.

2. – Description of the meson.

The meson is described by a field operator $\varphi_u(\boldsymbol{x})$ (where $\boldsymbol{x}$ are the space coordinates). $u = 1, 2, 3$, corresponds to the three charge states of the meson. φ_u transforms like a scalar under rotations and therefore describes a particle of spin 0 in accordance with experiment. In terms of destruction operators $a_u(\boldsymbol{k})$ and creation operators $a_u^+(\boldsymbol{k})$ one has

$$\varphi_u(\boldsymbol{x}) = (2\pi)^{-\frac{3}{2}}\int\frac{\mathrm{d}\boldsymbol{k}}{\sqrt{2\omega}}\left(a_u(\boldsymbol{k})\exp\left[i\boldsymbol{k}\boldsymbol{x}\right] + a_u^+(\boldsymbol{k})\exp\left[-\,i\boldsymbol{k}\boldsymbol{x}\right]\right), \tag{2}$$

where $\boldsymbol{k}$ is the momentum of the meson, $\omega = {}_{+}\sqrt{k^2+\mu^2}$ its frequency or energy ($\hbar = c = 1$) and μ is its rest mass. The operators a and a^+ satisfy the commutation relations

$$[a_u(\boldsymbol{k})a_v(\boldsymbol{k}')] = 0\,, \qquad [a_u(\boldsymbol{k})a_v^+(\boldsymbol{k}')] = \delta_{uv}\delta(\boldsymbol{k}-\boldsymbol{k}')\,, \tag{3}$$

and the energy of the field is given by

$$H_\pi = \sum_u\int \mathrm{d}^3k\,\omega a_u^+(\boldsymbol{k})\,a_u(\boldsymbol{k})\,. \tag{4}$$

It is easily shown, that on the strength of the commutation relations (3) one can define 3 operators T_u which commute with H_π

$$T_u = -i \sum_{vw} \varepsilon_{uvw} \int \mathrm{d}^3k\, a_v^+(\boldsymbol{k})\, a_w(\boldsymbol{k})\,, \tag{5}$$

where ε_{uvw} is the Ricci-symbol. The T_u do not commute with one another but satisfy the commutation relations

$$[T_1 T_2] = iT_3 \quad \text{and} \quad \text{cycl}\,, \tag{6}$$

identical with those for the angular momentum operators. T_3 can be interpreted as the meson charge. The operators

$$\alpha_\pm = \frac{1}{\sqrt{2}}(a_1 \mp ia_2)\,, \qquad \alpha_0 = a_3\,, \tag{7}$$

then describe the destruction of respectively positive negative and neutral mesons.

3. – The interaction.

The Hamiltonian of the interaction is defined by the following requirements:

1) It should be simple in form.
2) It should conserve angular momentum.
3) It should conserve parity if the meson is assumed to be pseudoscalar.
4) It should conserve isotopic spin.

By the latter requirement we mean that the operator of the interaction energy should commute with the total isotopic spin defined as

$$I_u = T_u + \tfrac{1}{2}\tau_u\,. \tag{8}$$

The requirement of simplicity has the following meaning. We may classify the interaction energies by the number of powers of the amplitude of the meson field which they contain, thus: $H' = H_1 + H_2 + \dots$ It is clear that H_1 is then completely defined by the requirements 2), 3) and 4), thus

$$H_1 = \sqrt{4\pi}\cdot\frac{f}{\mu}\int \mathrm{d}^3\boldsymbol{x}\,\tau_u\big(\sigma\cdot\mathrm{grad}\,\varphi_u(\boldsymbol{x})\big)\,v(\boldsymbol{x})\,, \tag{9}$$

where f is a dimensionless constant and $v(x)$ is a normalized function

$$\int v(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} = 1\,, \tag{10}$$

which — because of the requirement 3) depends only on $x = |\boldsymbol{x}|$.

H_2 admits a far greater variety. It will in general involve « bilocal » form factors of the nucleon. Though for a realistic description of meson nucleon scattering it is doubtlessly necessary to consider such terms (we know that there is S-wave scattering, but (9) only gives P-wave scattering) we shall not deal with them here.

$v(x)$ is called the nucleon form factor. It gives an idea of the extension of the region in which nucleons and mesons can interact. It is expected to extend over a radius of $\sim 1/M$, where M is the mass of the nucleons.

4. – Meson partial wave analysis.

It is clear from an inspection of equation (9) that the interaction energy H_1 will only describe processes in which P-state mesons are created or destroyed. All the other angular momentum states are totally unaffected by (9). In the following discussion we shall therefore only consider P-state mesons. It is readily seen that the operators

$$a_{iu}(\omega) = i\sqrt{\frac{3\omega}{4\pi k}}\int \mathrm{d}\Omega\, k_i\, a_u(\boldsymbol{k})\,, \tag{11}$$

can only destroy P-state mesons. ($\int\mathrm{d}\Omega$ indicates integration over all the directions of the vector $\boldsymbol{k}$, *i.e.* $\int \mathrm{d}^3k = \int k^2\,\mathrm{d}k \int\mathrm{d}\Omega$.) Using the commutation relations (3) one immediately finds that the non vanishing commutators of the new operators $a_{iu}(\omega)$ and $a^+_{iu}(\omega)$ are

$$[a_{iu}(\omega),\, a^+_{i'u'}(\omega')] = \delta_{ii'}\,\delta_{uu'}\,\delta(\omega - \omega')\,, \tag{12}$$

and one also obtains immediately for the P-wave contribution to the free Hamiltonian

$$H_0 = \sum_{iu}\int \mathrm{d}\omega\, a^+_{iu}(\omega)\, a_{iu}(\omega)\,\omega\;. \tag{13}$$

Angular moment and isotopic spin are given respectively by

$$iM_i = \sum_{klu} \varepsilon_{ikl}\int \mathrm{d}\omega\, a^+_{ku} a_{lu}\,, \qquad iT_u = \sum_{ivw} \varepsilon_{uvw}\int \mathrm{d}\omega\, a^+_{iv} a_{iw}\;. \tag{14}$$

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It is therefore seen that the canonical transformation (11) results in a complete symmetrization between the angular momentum and isotopic spin variables of the meson-field. This symmetry is an essential property of the model and holds also for the interaction energy. For, inserting from equation (2) into (9) and using the definition (11) we get for the interaction energy

$$H_1 = \int \mathrm{d}\omega \, V'_{iu}(\omega)\left(a_{iu}(\omega) + a^+_{iu}(\omega)\right) , \tag{15}$$

where $V'_{iu}(\omega)$ is defined as

$$V'_{iu} = \frac{f}{\mu} \sigma_i \tau_u \varrho(k) \sqrt{\frac{k^3}{3\pi}} , \tag{16}$$

and $\varrho(k)$ is the Fourier transform of the form factor of the nucleon:

$$\varrho(k) = \int v(x) \exp[i\boldsymbol{kx}] \, \mathrm{d}^3 x . \tag{17}$$

If the source extends over a distance $1/M$, ϱ will be ~ 1 for $k \ll M$ and will drop off to zero for momenta of the order M.

5. – The Hilbert space.

Since all except the P-states are « quiescent » in the present model, we shall only consider that section of the Hilbert space which is formed from the four states χ_α of the source (we abbreviate the notation by putting $s, t = \alpha$) and all the possible states of P-wave mesons. The generic vector of this section of the Hilbert space is then given by

$$\chi_\alpha(1, \ldots n) = \frac{1}{\sqrt{n!}} a^+(1) \ldots a^+(n) \chi_\alpha , \tag{18}$$

where the numbers $1, \ldots n$ in the argument of the operators a^+ stand short for the parameters $i_1, u_1, \omega_1, \ldots i_n, u_n, \omega_n$. The states with $n = 0$ are the states of the bare source, with the properties

$$H_0 \chi_\alpha = 0 , \qquad a_{iu}(\omega) \chi_\alpha = 0 . \tag{19}$$

A typical state (18) represents n mesons in the states characterized by the indices $1, \ldots n$ and the source in the state characterized by the index α.

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The states (18) are eigenstates of the « free » Hamiltonian H_0:

$$H_0\chi_\alpha(1 \dots n) = \chi_\alpha(1 \dots n) \sum_{i=1}^{n} \omega_i\,, \tag{20}$$

they are in general not eigenstates of I_3 and $J_3 = M + \frac{1}{2}\sigma_3$.

6. – The physical nucleon.

The real nucleon will be represented by a state vector φ_α, which in general will be a superposition of vectors of the type (18). φ_α will be an eigenstate of H. It is subject to the limiting condition

$$\lim_{f=0} \varphi_x = \chi_\alpha\,, \tag{21}$$

which expresses, that in the absence of interaction there is no difference between a physical and a bare nucleon. The eigenvalue E_s of the Hamiltonian defined by

$$(H - E_s)\varphi_x = 0\,, \tag{22}$$

is called the self energy of the source. Since also the interaction energy conserves angular momentum and isotopic spin it is clear that the states φ_x will be degenerate, because the states χ_α are degenerate.

To obtain formal expressions for the self energy as well as for φ_α we introduce a projection operator Λ with the property

$$\Lambda\chi_\alpha = 0\,, \qquad \Lambda\chi' = \chi'\,, \tag{23}$$

where χ' is any state (18) other than the bare nucleon state. One can then write for φ_α

$$\varphi_x = N\chi_\alpha + \Lambda\varphi_\alpha\,. \tag{24}$$

This follows from the fact that χ_β with $\beta \neq x$ cannot appear in the expansion of φ_x. For all the terms making up φ_α must transform in the same fashion as χ_x under rotations and isotopic spin transformations. N is given by

$$N = (\chi_\alpha\varphi_\alpha)\,, \tag{25}$$

and represents the probability amplitude for finding the real nucleon in its bare state. Multiplying equation (22) by Λ from the left, remembering that Λ

commutes with H_0 and using equation (24) we obtain

$$(H_0 + \Lambda H_1 - E_s)\Lambda\varphi_a = -\Lambda H_1 \chi_\alpha N\,, \tag{26}$$

where on the right hand side we can also drop the operator Λ since $\chi H'\chi = 0$. We may therefore solve (31) formally by writing

$$\Lambda\varphi_\alpha = -N(H_0 + \Lambda H_1 - E_s)^{-1} H_1 \chi_\alpha\,, \tag{27}$$

which is the familiar perturbation expansion, with the difference that the energy appearing in the denominators is the energy of the real nucleon and not that of the bare nucleon. Correspondingly the energy of the physical nucleon state does not appear as the result of a power series expansion but as the solution of an implicit equation. This equation is obtained by multiplying (22) by χ from the left. Using (27) one gets the secular equation

$$f(E_s) \equiv E_s + \chi H_1 (H_0 + \Lambda H_1 - E_s)^{-1} H_1 \chi = 0\,. \tag{28}$$

One can now quite easily show from (28) that this equation has one and only one negative root

$$M = -E_s > 0\,. \tag{29}$$

For differentiating (28) we obtain

$$\frac{\partial f}{\partial E} = 1 + \chi H_1 (H_0 + H_1 \Lambda - E)^{-1} (H_0 + \Lambda H_1 - E)^{-1} H_1 \chi\,, \tag{30}$$

and it is clear that the right hand side is always $\geqslant 1$. Since asymptotically $f(E) = E$ for $E = \pm\infty$ it follows that there must be one root of the type (28) in the region of the discrete spectrum. By expanding (28) into a power series in H, one can also see that there are no solutions in $-M < E < \mu - M$.

Mass renormalization consists simply in replacing H by $H + M$. The real nucleon has then $E = 0$, *i.e.*, the same energy as the bare nucleon. It has been assumed — though not proven — that after mass renormalization the spectrum of the renormalized Hamiltonian $H + M$ coincides with that of the free Hamiltonian and is given by equation (20).

7. – The scattering of mesons by nucleons.

In the fixed source theory all information about P-state scattering is derived from a wave function $\psi_{j\alpha}(\omega)$, solution of the Schrödinger equation

$$(H - \omega)\psi_{j\alpha}(\omega) = 0\,, \tag{31}$$

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and satisfying the limiting condition

$$\lim_{f\to 0} \psi_{j\alpha}(\omega) = a_j^+(\omega)\chi_\alpha \,. \tag{32}$$

The « spectral hypothesis » ensures that such a solution should exist. To obtain a formal expression for this solution we put

$$\psi_{j\alpha}(\omega) = a_j^+(\omega)\varphi_\alpha + X_{j\alpha}(\omega)\,, \tag{33}$$

and because of (32) expect X to vanish for $f \to 0$. The first term of (33) gives a better approximation to the asymptotical physical conditions than the limit in (32), since in (33) all our information about the real nucleon is incorporated. Remembering that because of mass renormalization we have $H\varphi_\alpha = 0$ and that therefore

$$Ha_j^+(\omega)\varphi_\alpha = [Ha_j^+(\omega)]\varphi_\alpha = \omega a_j^+(\omega)\varphi_\alpha + V_j(\omega)\varphi_\alpha\,,$$

we obtain for X

$$(H-\omega)X_{j\alpha}(\omega) + V_j(\omega)\varphi_\alpha = 0\,.$$

The matrix multiplying X from the left is singular, since ω is one of the eigenvalues of H. This singularity can be formally removed by adding a small imaginary part to H. This gives one two typical solutions

$$\psi_{j\alpha}^{(\pm)}(\omega) = \big(a_j^+(\omega) - (H-\omega \mp i\eta)^{-1}\,V_j(\omega)\big)\varphi_\alpha\,, \tag{34}$$

in which η is to be regarded positive but infinitesimal. The physical significance of the solutions (34) is understood by considering wavepackets of the form

$$F_{j\alpha}^{(\pm)}(\omega_0, t) = \int \mathrm{d}\omega\, f(\omega\omega_0)\psi_{j\alpha}(\omega)\exp[-i\omega t]\,,$$

where $f(\omega\omega_0)$ is a weight function, centered around ω_0. Carrying out the ω integration it is then seen that the wave packet coming from the second term in (34) vanishes in the limit $t = -\infty$ for $\psi^{(+)}$ and in the limit $t = \infty$ for $\psi^{(-)}$ It follows that asymptotically we have

$$F_{j\alpha}^{(\pm)}(\omega_0; \mp\infty) = a_j^+(\omega_0)\exp[-i\omega_0 t]\varphi_\alpha\,. \tag{35}$$

This equation is valid for infinitely narrow wave packes $f(\omega, \omega_0) \to \delta(\omega-\omega_0)$. Since the normalization is completely defined by the asymptotic behaviour of the wave functions and since asymptotically we have conservation of energy, it follows that in order to determine the norm of ψ we have to calculate the singular part of the norm of $a^+_j(\omega)\varphi_\alpha$. This gives because of the commutation relations (12)

$$(\psi^{(+)}_{j'\alpha'}\omega')\psi^{(+)}_{j\alpha}(\omega)) = \delta(\omega'-\omega)\,\delta_{j'j}\delta_{\alpha'\alpha}\,. \tag{36}$$

The S-matrix is defined as the overlap between ingoing and outgoing solutions, thus

$$(\psi^{(-)}_{j'\alpha'}(\omega')\psi^{(+)}_{j\alpha}(\omega)) = \delta(\omega-\omega')\,(j'\alpha'\,|\,S(\omega)\,|\,j\alpha)\,. \tag{37}$$

It represents the probability amplitude for finding the system in the state $j'\alpha'$ at time $t=\infty$ provided that the same system has been found to be in the state j, α at time $t=-\infty$. Using the obvious relation

$$2\pi i\,\delta(x) = \frac{1}{x-i\eta} - \frac{1}{x+i\eta} \tag{38}$$

we may write formally for (34)

$$\psi^{(+)}_{j\alpha}(\omega) = \psi^{(-)}_{j\alpha}(\omega) - 2\pi i\,\delta(H-\omega)\,V_j(\omega)\varphi_\alpha\,, \tag{39}$$

and multiplying by $\psi^{(-)}_{j'\alpha'}(\omega')$ from the left, we obtain for the scattering elements of the S-matrix

$$(j'\alpha'\,|\,S(\omega)\,|\,j\alpha) = \delta_{jj'}\,\delta_{\alpha\alpha'} - 2\pi i(\psi_{j'\alpha'}(\omega)\,V_j(\omega)\varphi_\alpha)\,. \tag{40}$$

Above the two meson threshold new processes become possible. If $\varphi^{(-)}_n$ is a general set of wave functions of the ingoing type it is obvious that we can generalize (40) to

$$(n\,|\,S(\omega)\,|\,j\alpha) = I - 2\pi i(\varphi^{(-)}_n\,V_j(\omega)\varphi_\alpha)\,, \tag{41}$$

where of course the energy of the state φ_n must equal ω. $(n\,|\,S(\omega)\,|\,j,\alpha)$ is therefore the probability amplitude of finding the system in the state n at time $t=\infty$ provided that it was in the state j, α at time $t=-\infty$. Conservation of probability requires that

$$S^+S = I\,. \tag{42}$$

The Low equation which we are going to discuss concerns the expression appearing on the right hand side of equation (41). This quantity has the property that its behaviour for $E_n \neq \omega$ is completely determined by its value

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on the « energy shell ». For because of (16) we can put

$$(\varphi_n^{(-)}\,V_j(\omega)\varphi_\alpha) = k^{\frac{3}{2}}\varrho k_n^{-\frac{3}{2}}\varrho_n^{-1}(n\,|\,T(E_n)\,|\,j\alpha)\,, \tag{43}$$

where $k_n = \sqrt{E_n^2 - \mu^2}$. Obviously one has because of (40) and (41)

$$(n\,|\,S(E_n)\,|\,j\alpha) = I - 2\pi i(n\,|\,T(E_n)\,|\,j\alpha)\,. \tag{44}$$

The T matrix can therefore — apart from a factor $-2\pi i$ — be identified with the Heisenberg R-matrix.

To get the Low equation we now remember that it follows from (34) that

$$\psi_{j'\alpha'}^{(-)*}(\omega) = \varphi_{\alpha'}^{*}a_{j'}(\omega) - \sum_n (\varphi_{\alpha'}V_{j'}(\omega)\varphi_n^{(-)})\,\frac{1}{E_n - \omega - i\eta}\,\varphi_n^{(-)*}\,. \tag{45}$$

Multiplying this equation from the right by $V_j(\omega)\varphi_\alpha$ and remembering (43) we obtain

$$(j'\alpha'\,|\,T(\omega)\,|\,j\alpha) = (\varphi_{\alpha'}\,a_{j'}\,(\omega)V_j(\omega)\varphi_\alpha) - \sum_n \frac{(\varphi_{\alpha'}V_{j'}(\omega)\varphi_n^{(-)})(\varphi_n^{(-)}V_j(\omega)\varphi_\alpha)}{E_n - \omega - i\eta}\,. \tag{46}$$

Now since the a and V commute, we may put the $a_{j'}$ into contact with φ_α. But

$$Ha'\varphi_\alpha = -\,\omega'a'\varphi_\alpha - V'\varphi_\alpha\,,$$

where a' V' are short for $a_{j'}$ $V_{j'}$ and therefore

$$a'\varphi_\alpha = (H + \omega)^{-1}V'\varphi_\alpha\,,$$

(there is no ambiguity in the matrix $(H+\omega)^{-1}$, since all the eigenvalues of H are positive — after mass renormalization). Inserting into equation (46) and using (43) we get the Low equation in the form

$$(j'\alpha'\,|\,T(\omega)\,|\,j\alpha) = -\sum_n \left\{\frac{(n\,|\,T(E_n)\,|\,j\alpha')^*(n\,|\,T(E_n)\,|\,j'\alpha)}{E_n + \omega} + \frac{(n\,|\,T(E_n)\,|\,j'\alpha')^*(n\,|\,T(E_n)\,|\,j\alpha)}{E_n - \omega - i\eta}\right\}\frac{k^3\varrho^2}{k_n^3\varrho_n^2}\,. \tag{47}$$

We shall discuss this equation in the next section. Here we only remark that this integral equation only involves quantities defined on the energy shell.

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8. – Crossing symmetry and conservation of probability.

It follows from an inspection of the Low equation that its solutions must show the following functional symmetry:

$$(j\alpha' \,|\, T(\omega) \,|\, j'\alpha) = (j'\alpha' \,|\, T(-\omega) \,|\, j\alpha)\,, \tag{48}$$

which is called crossing symmetry and which allows one to continue the S-matrix into the non-physical region of negative frequencies. To derive (48) one makes use of the fact that in the denominator of the first term of (47) one can replace E_n by $E_n - i\eta$, because the integral has no pole. One has also to require that $k(-\omega) = k(\omega)$, which limits the applicability of (48) to one leaf of the ω-plane divided by two cuts ranging from $-\infty$ to $-\mu$ and from μ to ∞.

The conservation of probability — the optical relation — is expressed by

$$(j'\alpha' \,|\, T(\omega) - T^{+}(\omega) \,|\, j\alpha) = -2\pi i \sum_n \delta(E_n - \omega)\,(n \,|\, T(E_n) \,|\, j'\alpha')^{*}\,(n \,|\, T(E_n) \,|\, j\alpha)\,, \tag{49}$$

which follows from (47) by subtracting its Hermitian conjugate and using (38). This equation is completely equivalent to the relation $S^{+}S = I$ (the unit matrix being defined on the energy shell) which expresses the conservation of probability.

9. – Linearization of the Low equation.

The optical relation (49) allows one to linearize Low's equation. On the right hand side of equation (47) the sum is extended over all the physical states of the system. It will therefore consist of a contribution ($n = 0$) of the ground state of the source and a contribution from the continuum. Calling the first F and applying to equation (47) the obvious relation

$$\sum_n{}' f(E_n) = \int_{\mu}^{\infty} \mathrm{d}\omega' \sum_n{}' \delta(E_n - \omega') f(E_n)\,, \qquad \sum{}' = \sum_{n \neq 0}\,,$$

valid for a continuum of eigenvalues $\mu \leqslant E_n \leqslant \infty$ it immediately follows from (49) that

$$(j'\alpha' \,|\, T(\omega) \,|\, j\alpha) = F + \int \mathrm{d}\omega' \, \frac{k^3 \varrho^2}{k'^3 \varrho'^2} \cdot \frac{1}{2\pi i} \left\{ \frac{(j'\alpha' \,|\, T(\omega') - T^{+}(\omega') \,|\, j\alpha)}{\omega' - \omega - i\eta} + \frac{(j'\alpha' \,|\, T(\omega') - T^{+}(\omega') \,|\, j'\alpha)}{\omega' + \omega} \right\}, \tag{50}$$

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with F defined by

$$F = \frac{1}{\omega} \sum_{\beta} \left[(\varphi_\beta V_{j'}(\omega)\varphi_{\alpha'})^* (\varphi_\beta V_j(\omega)\varphi_\alpha) - (\varphi_\beta V_j(\omega)\varphi_{\alpha'})^* (\varphi_\beta V_{j'}(\omega)\varphi_\alpha) \right], \tag{51}$$

F will be determined in the next section and we shall then show that it is Hermitian in the indices j, α. It is then immediately seen that the imaginary part of equation (51) is an identity. Putting

$$T = -\frac{k^3\varrho^2}{\pi}\,(A(\omega) + iB(\omega)) \qquad A^+(\omega) = A(\omega) \qquad B^+(\omega) = B(\omega)\,, \tag{52}$$

(we have chosen the numerical factors in such a way that the operators A and B defined in $j\alpha$ space allow a simple interpretation in terms of the scattering phases) and remembering that

$$P\!\int \frac{\mathrm{d}\omega'}{\omega' - \omega} = \frac{1}{2}\left(\int \frac{\mathrm{d}\omega'}{\omega' - \omega - i\eta} + \int \frac{\mathrm{d}\omega'}{\omega' - \omega + i\eta} \right),$$

where P indicates the principal part of the integral, we get from equation (51)

$$A(\omega) = -\frac{F\pi}{k^3\varrho^2} + \frac{1}{\pi}\,P\!\int \mathrm{d}\omega' \left(\frac{B(\omega')}{\omega' - \omega} + \frac{B^*(\omega')}{\omega' + \omega} \right), \tag{53}$$

B^* is defined by

$$(j'\alpha'|B^*(\omega)|j\alpha) = (j\alpha'|B(\omega)|j'\alpha)\,. \tag{54}$$

We note that equation (50) only involves the elements of the T-matrix on the energy shell. It therefore represents a relation between quantities which are directly observable. If the imaginary part of T is known, the real part is defined by equation (53). The structure of (53) is therefore the same as that of the well known dispersion relations, which define the real part of the scattering matrix in terms of its imaginary part, which is related by the optical equation to the total cross section.

10. – Determination of F.

It follows from the symmetry of our problem that there must exist a number x such that

$$(\varphi_\beta V_j(\omega)\varphi_\alpha) = x(\chi_\beta V_j(\omega)\chi_\alpha)\,. \tag{55}$$

For under any spin or isotopic spin rotation the φ and the χ transform in the same way. That x is real follows by taking the complex conjugate of (54). We can therefore write for F (remembering (16))

$$F = \frac{x^2}{\omega}[V_{j'}V_j] = \frac{f_r^2}{3\pi\mu^2\omega}\, k^3\varrho^2[\sigma_{i'}\,\tau_{u'},\sigma_i\tau_u]\,, \tag{56}$$

where the V_j are interpreted as operators acting on the vectors χ. f_r — the so called renormalized coupling constant — is defined as fx. Remembering that

$$\sigma_i\sigma_i = i\varepsilon_{i'ik}\sigma_k + \delta_{i'i}\,, \qquad \tau_{u'}\tau_u = i\varepsilon_{u'uv}\tau_v + \delta_{u'u}\,,$$

we immediately find that we may write for F

$$F = \frac{2f_r^2}{3\pi\mu^2\omega}\, k^3\varrho^2[\delta_{i'i}i\varepsilon_{u'uv}\tau_v + \delta_{u'u}i\varepsilon_{i,ik}\sigma_k]\,.$$

This can be put into an even shorter form by remembering equation (14) from which it follows that $(i'|M_k|i) = -i\varepsilon_{i'ik}$ and therefore:

$$F = -\frac{f_r^2}{3\pi\mu^2\omega}\, k^3\varrho^2\big\{(\sigma M) + (\tau T)\big\}, \tag{57}$$

where the round brackets ( ) indicate the scalar product. This expression suggests to write both A and B in operator form. Using an argument similar to that used in the first lecture of Prof. ASHKIN we can deduce that these operators can be either independent of σ and τ or at most linear in either of them. (Higher powers of σ and τ can always be reduced to the 0-th or first by the use of the commutation relations for these operators.) Since σ and τ are vector quantities and since the theory is symmetrical under spin and isotopic spin rotations, since further—as we have already seen—the theory is invariant against an exchange between σ and τ the most general form of A and B (valid for one nucleon plus one meson initial and final states) must be

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + ((\sigma M) + (\tau T))\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} + (\sigma M)(\tau T)\begin{pmatrix} a_3 \\ b_3 \end{pmatrix}, \tag{58}$$

$a_1 \dots b_3$ are real functions of ω and the crossing theorem (48) requires that

$$\begin{pmatrix} a_{1,3}(-\omega) \\ b_{1,3}(-\omega) \end{pmatrix} = \begin{pmatrix} a_{1,3}(\omega) \\ b_{1,3}(\omega) \end{pmatrix} \qquad \begin{pmatrix} a_2(-\omega) \\ b_2(-\omega) \end{pmatrix} = -\begin{pmatrix} a_2(\omega) \\ b_2(\omega) \end{pmatrix}, \tag{59}$$

since both M and T change sign under crossing. Inserting (58) into the Low equation (52) we obtain the following set of equations

$$a_i(\omega) = \frac{2f_r^2}{3\mu^2\omega}\,\lambda_i' + \frac{P}{\pi}\int\limits_\mu^\infty d\omega'\left(\frac{b_i(\omega')}{\omega'-\omega} + (1-2\lambda_i')\,\frac{b_i(\omega')}{\omega'+\omega}\right), \tag{60}$$

λ' is the vector $(0, 1, 0)$. It is seen that the f_r^2 term only concerns the amplitudes a_2 and b_2.

11. – Low equation and dispersion relations.

Equation (60) formally resembles a dispersion relation. In this section I want to show that it is indeed a dispersion relation. I first consider the case of $i = 1, 3$. In this case equations (60) may be written

$$a(\omega) = \frac{2}{\pi}\int\limits_\mu^\infty d\omega'\,\frac{\omega' b(\omega')}{\omega'^2 - \omega^2}\,. \tag{61}$$

We now consider the function $f(z)$ defined by

$$f(z) = \frac{2}{\pi}\int\limits_\mu^\infty d\omega'\,\frac{\omega' b(\omega')}{\omega'^2 - z^2}\,. \tag{62}$$

This function has the following properties:

1) it vanishes for $z \to \infty$ at least as const$/z^2$, provided that $\int \omega' b(\omega')$ exists;

2) it has a line of discontinuity extending on the real axis from $-\infty$ to $-\mu$ and from μ to $+\infty$;

3) in the z-plane cut by the lines along the real axis $f(z)$ is a regular function;

4) $f(z)$ is real for real $-\mu < \omega < \mu$;

5) using (38) it is immediately seen that for real ω and $|\omega| \geqslant \mu$ we have

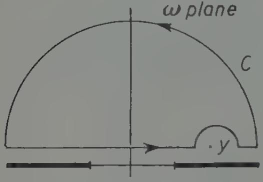
$$\lim_{\eta \to +0} f(\omega + i\eta) = a(\omega) + i\theta(\omega) b(\omega)\,, \tag{63}$$

where $\theta(\omega) = \pm 1$ for $\omega \gtrless 0$ is a step function. For positive frequencies f therefore coincides with the scattering amplitude;

6) f has the following symmetry property:

$$f^*(-z) = f(z)\,. \tag{64}$$

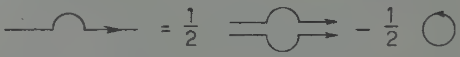
Since $f(z)$ is regular in the upper half plane it must obviously satisfy Cauch'sy relation and we have



$$\int_C \frac{f(z)}{z-y} = 0\,,$$

where the path C is indicated in the drawing and y is a point situated in the upper half of the z-plane with an infinitesimal imaginary part. Since f vanishes sufficiently rapidly because of 1) we may neglect the contribution of the semicircle and use the symbolical equation

(65)



the first part represents the principal value, so that we have

$$P\!\int \frac{f(z)\,\mathrm{d}z}{z-y} = \pi i f(y) \qquad \text{for } y = \text{real}: \tag{66}$$

Adding the same equation for y replaced by $-y$ we immediately obtain by using the symmetry condition (64)

$$f(y) + f^*(y) = \frac{P}{i\pi}\int z\,\mathrm{d}z\,\frac{f(z)-f^*(z)}{z^2-y^2}\,. \tag{67}$$

But this is completely equivalent to (61), provided that $\operatorname{Im} f = 0$ for $-\mu < \omega < \mu$. We have therefore shown that the Low equations for $i = 1, 3$ can be obtained in the following way.

Let a and b be the real and imaginary part of the scattering matrix divided according to (52) by the square of the form-factor. We then require that the analytical continuation of $f(\omega)$ defined by (63) converge sufficiently rapidly at infinity, have a vanishing imaginary part for $-\mu < \omega < \mu$ and be regular in the upper half plane. The proof is quite similar for $i = 3$, only that in this case $f(z)$ must have a pole at $z = 0$, and that (63) has to be replaced by $f^*(z) = -f(-z)$.

An important difference between the dispersion relations obtained in a relativistic theory by means of the causality and locality requirement and the dispersion relations (61) derived here, is that in a relativistic theory the

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function which obeys the dispersion relations coincides with the elements of the T-matrix in the physical region of the ω-plane. Here, we have derived them for the quantity T/ϱ^2. The appearance of the form-factor is a serious difficulty because it renders all assumptions about the asymptotic behaviour of the scattering amplitudes extremely doubtful. Since, however, no physical significance can be attributed to the amplitudes determined from the fixed source theory and evaluated in the limit of energies tending to infinity, the difficulty becomes just one other aspect of the dilemma of dispersion relations: if we try to compare dispersion relations with experiment, we may fail, because such a comparison requires some information about the behaviour of cross-sections at infinite energies (right hand side of dispersion relations) which experimenters still obstinately refuse to supply.

12. – Phase analysis.

The operators (σM) and (τT) have values which are completely defined in terms of the total angular momentum J, and the total isotopic spin I. Either of these variables can have values $\frac{3}{2}$ or $\frac{1}{2}$ and one has $(\sigma M)=+1$ for $J=\frac{3}{2}$ and $(\sigma M)=-2$ for $J=\frac{1}{2}$. We shall denote the four amplitudes correposuding to the combinations $(J, I)=(11, 13, 31, 33)$ by the indices 1, 2, 3 respectively. (Only three values are necessary since it follows from the symmetry between spin and isotopic spin that the scattering matrices 31 and 13 are the same). We can therefore write

$$(68)\qquad \left\{\begin{aligned} A_1 &= a_1-4a_2+4a_3\\ A_2 &= a_1-a_2-2a_3\\ A_3 &= a_1+2a_2+a_3 \end{aligned}\right\}\qquad \text{and similar equations for } B \text{ and } b.$$

or also more briefly

$$(69)\qquad A_k=U_{ki}a_i\,,$$

(summation convention). U_{ki} is the matrix of coefficients of equation (68). Multiplying (60) by U_{ki} from the right, we can obtain the dispersion relations for A in the form:

$$(70)\qquad A_k(\omega)=\lambda_k\frac{f_r^2}{\mu^2\omega}+\frac{1}{\pi}P\!\int\!\mathrm{d}\omega'\left(\frac{B_k(\omega')}{\omega'-\omega}+X_{kl}\,\frac{B_l(\omega')}{\omega'+\omega}\right).$$

Here λ_k is defined by

$$(71)\qquad \lambda_k=\frac{2}{3}\,U_{ki}\lambda'_i=\frac{1}{3}\begin{pmatrix}-8\\-2\\+4\end{pmatrix},$$

and the so called crossing matrix X is given by

$$X_{kl} = U_{ki}(1 - 2\lambda_i')U_{il}^{-1}.$$

One easily verifies that, since

$$U^{-1} = \begin{pmatrix} 1 & 4 & 4 \\ -1 & -1 & 2 \\ 1 & -2 & 1 \end{pmatrix} \cdot \frac{1}{9},$$

X must be given by

$$X = \begin{pmatrix} 1 & 8 & 16 \\ -2 & 7 & 4 \\ 4 & 4 & 1 \end{pmatrix} \cdot \frac{1}{9}. \tag{73}$$

For energies for which no inelastic processes (like the conversion of 1 meson into two) are possible, the quantities A are directly related to the phases. For in this case one has $S_k = \exp[2i\delta_k]$ and therefore

$$k^3\varrho^2 A_k(\omega) = \tfrac{1}{2}\sin 2\delta_k\,, \qquad B_k(\omega)k^3\varrho^2 = -\tfrac{1}{2}\cos 2\delta_k + \tfrac{1}{2}\sin^2\delta_k\,. \tag{65}$$

It must however be noted that by the very nature of the equations, the right-hand side of (70) will always contain contributions from these inelastic processes.

Time does not allow that I discuss the applications of equation (70) to the actual problem of meson nucleon scattering: this will be done later in this course. Let me, however, conclude by pointing out one important feature of this equation. The term with f_r^2 in (70) gives a negative contribution to A for the index combinations (11, 13, 31) and a positive contribution for the amplitude describing the $\frac{3}{2}\frac{3}{2}$-state (equation (71)). Since a positive phase at the threshold describes an attractive force between the particles, we conclude that it is very likely that meson nucleon forces at threshold are attractive only in the $\frac{3}{2}\frac{3}{2}$-state an hypothesis amply borne out by experiment. Its mathematical deduction from equation (70) is in my opinion not free from additional hypotheses and I hope that we shall hear more about it in the course of these lectures.

REFERENCES

[1] G. F. CHEW: *Phys. Rev.*, **91**, 1748, (1954); **91**, 1755, (1954).
[2] G. F. CHEW and F. E. LOW: *Phys. Rev.*, **101**, 1570, (1956).
[3] G. C. WICK: *Rev. Mod. Phys.*, **27**, 339, (1955).
[4] F. E. LOW; *Rev. Mod. Phys.*, **29**, 216, (1957).

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SUPPLEMENTO AL VOLUME XIV, SERIE X
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Low Energy Pion Physics.

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1. – Introduction.

In these lectures we are going to take the pion (π), the nucleon (N), and to some extent the photon (γ), and talk about them in isolation. With the use of phenomenological theory we may build up classical pion physics as a closed subject. This is very much a historical procedure which will certainly be seen to be of limited value when some real understanding of the fundamental particles is achieved.

We define a low energy pion physics which comprises:

1) Reactions at low (positive) energy.

Consider, for example,

(pion-nucleon scattering in the C.M. system).

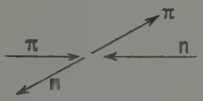

Fig. 1.

All reactions take place in s-states and are therefore characterized by an isotropic distribution of the products apart from Coulomb effects at small angles and low energies.

2) π^- capture in hydrogen or deuterium.

The π^- is stopped, passes down through the optical levels until it reaches the K-shell, and then in due course absorption is seen for reactions in which energy is released. We can discuss the lifetime for the absorption processes.

3) The π^- may be captured by a heavier nucleus. Here the energy released from the optical transitions is sufficient to enable us to observe the X-rays produced. We may calculate the exact energy of these assuming a purely electromagnetic interaction, and then the shift in energy of the observed X-rays gives us information on the pion-nucleon interaction. The lifetime for the processes here is obtained from the width of the spectral line *i.e.* the lifetime may be inferred from the energy uncertainty.

We shall for the moment concentrate on 1) and 2) and list the reactions in the following manner (*).

$$\begin{tikzcd}
 & & & \sigma_s(p+p \to \pi^+ + d) \arrow[d, dash, "c.s."] & \\
 & & & {[\sigma_s(n+n \to \pi^- + d)]} \arrow[d, dash, "d.b."] & \\
 & \text{zero energy processes} \arrow[d] & & {[\sigma_s(\pi^- + d \to n+n)]} \arrow[d, dash, "\sigma \to \omega ."] & \\
\omega(\pi^- + d \to \gamma + 2n) \arrow[rrr, dash, "S."] \arrow[d, dash, "T."] & & & \omega(\pi^- + d \to n+n) & \\
\omega(\pi^- + p \to \gamma + n) \arrow[rrr, dash, "R."] \arrow[d, dash, "\sigma \to \omega"] & & & \omega(\pi^- + p \to \pi^0 + n) . \arrow[d, dash, "\sigma \to \omega"] & \\
{[\sigma_s(\pi^- + p \to \gamma + n)]} \arrow[d, dash, "d.b."] & & & \sigma_s(\pi^- + p \to \pi^0 + n) \arrow[dl, dash, "c.i."'] \arrow[dr, dash, "c.i."] & \\
{[\sigma_s(\gamma + n \to \pi^- + p)]} \arrow[d, dash] \arrow[dr, dash, "r_0"] & & \sigma_s(\pi^+ + p \to \pi^+ + p) & & \sigma_s(\pi^- + p \to \pi^- + p) \\
r = \dfrac{\sigma_s(\gamma + d \to \pi^- + 2p)}{\sigma_s(\gamma + d \to \pi^+ + 2n)} & \sigma_s(\gamma + p \to \pi^+ + n) . & & &
\end{tikzcd}$$

where [ ] indicates that a reaction has not been directly observed, R, is the Panofsky ratio, S the similar ratio in deuterium, σ a cross-section and ω a transition rate. The abbreviations on the connecting links will be explained in due course. It will be noticed that in the bottom left hand corner we have used the well known cross-section $\sigma_s(\gamma + p \to \pi^- + n)$, and the measured ratio r to obtain $\sigma_s(\gamma + n \to \pi^- + p)$.

In addition to these reactions we have the following:

$\sigma_s(p+p \to p+n+\pi^+)$,

$\sigma_s(p+p \to \pi^0 + p + p)$, very small,

$\sigma_s(\gamma + p \to \pi^0 + p)$, again very small,

which do not fit into the general scheme.

(*) The index s means s-wave processes.

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2. – Phenomenological theory.

We will now explain the connecting links which are as follows:

1) c.s. = charge symmetry.

2) c.i. = charge independence. This has already been discussed by Prof. Ashkin in his lecture on the isotopic spin formalism for a pion nucleon system. For very low energy scattering, which makes up the right hand corner entry to the diagram, we need only two parameters, namely α_3 - the phase shift corresponding to $I=\frac{3}{2}$, and α_1, that for $I=\frac{1}{2}$. For only two parameters we need only two experiments, but the third experiment serves as a check.

Now neglecting Coulomb effects:

$$\sigma_s(\pi^+\to\pi^+) = 4\pi\lambda\!\!\!^{-2}\alpha_3^2 = 4\pi\lambda\!\!\!^{-2}_c\left(\frac{\alpha_3}{\eta}\right)^2 = 4\pi\lambda\!\!\!^{-2}_c a_3^2\,,$$

where $\lambda\!\!\!^{-}_c$ is the pion Compton wave length, η the pion momentum in units of μc and $\lambda\!\!\!^{-}_c a_3$ is the scattering length.

$$\sigma_s(\pi^-\to\pi^-) = \frac{4\pi}{9}\,\lambda\!\!\!^{-2}_c(a_3+2a_1)^2\,,$$

$$\sigma_s(\pi^-\to\pi^0) = \frac{8\pi}{9}\,\lambda\!\!\!^{-2}_c(a_3-a_1)^2\left(\frac{v^0}{v^-}\right).$$

In the three expressions the factors: $1:\frac{1}{9}:\frac{2}{9}$ come from the Clebsch-Gordon coefficients used in the isotopic spin wave functions of the pion nucleon system, and v^0 is the velocity of the outgoing π^0 and v^- the velocity of the incoming π^-. The factor v^0/v^- is a kinematic factor to account for the π^- — π^0 mass difference which cannot be neglected at such low energies.

The scattering lengths may be thought of as constant at low energy, but it has been suggested recently that this is only true to a first order approximation (see Prof. Cini's lecture).

3) d.b. = detailed balancing.

We can incidentally obtain the pion spin from this.

Consider:

$$[\mathrm{n}+\mathrm{n}\rightleftharpoons\pi^++\mathrm{d}] \qquad \text{or} \qquad \underbrace{\mathrm{p}+\mathrm{p}}_{\text{system } A}\rightleftharpoons\underbrace{\pi^++\mathrm{d}}_{\text{system } B}$$

From the Fermi «Golden Rule» we have

$$\omega_{A\to B} = \frac{2\pi}{\hbar}|H_{BA}|^2\frac{dN_B}{dE},$$

$$\omega_{B\to A} = \frac{2\pi}{\hbar}|H_{AB}|^2\frac{dN_A}{dE}.$$

Now $|H_{AB}| = |H_{BA}|$, since one is the conjugate of the other.

$$\frac{\sigma_{A\to B}}{\sigma_{B\to A}} = \frac{dN_B/dE}{dN_A/dE}\,\frac{v_A}{v_B} = \frac{P_A^2}{P_B^2},$$

but we have not yet put in any spin factors. These are as follows: $(2S+1)$ for the pion, $(2J+1)=3$ for the deuteron which has spin $=1$, and $(2I+1)=2$ for each proton (spin $=\frac{1}{2}$)

$$\frac{\sigma_{A\to B}}{\sigma_{B\to A}} = \frac{P_B^2(2S+1)^3}{P_A^2\cdot 2^2\cdot\frac{1}{2}}.$$

The factor $\frac{1}{2}$ in the denominator comes in from the Pauli exclusion principle, since we have two protons and on the average half the spin states are excluded. We can digress to higher energies for a moment and examine the graph of

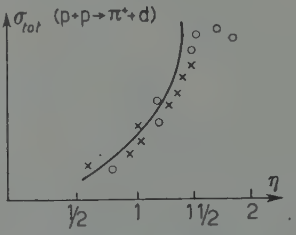

Fig. 2.

$\sigma_{\text{tot}}(p+p\to\pi^+ + d)$ *vs.* η,

where

$x = A\to B$,

$0 = B\to A$.

The solid curve shows a dependence on η^3 which is predicted for P wave production, and the good agreement between theory and experiment is obtained by putting the pion spin $S=0$. If we had $S\neq 0$ only an incredibly accidental polarization effect could give us the smooth experimental curve, and thus we here have good reason for believing the pion to be a spin zero particle.

The fact that the π^0 decays into 2 photons lends weight to the assumption of zero spin for the π-meson.

There are two types of experiment which cast doubt on this spin assignment

a) π-mesons were brought to rest in various visual detectors, and the subsequent angular distribution of the μ-mesons emitted was observed to be anisotropic. This means that either the spin of the π-meson is non-zero, or

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that the very general relation between angular momentum and rotations does not hold. However, there may have been an experimental bias, because of the visual technique employed.

b) A left-right asymmetry (~ 70/30) has been observed in the scattering of π^+-mesons off helium. If the π-meson has zero spin, this should not be observed.

It is to be hoped that these difficulties will be resolved in time.

It is to be noted that the fixed-source theory says nothing about the coupling of s-state pions. Even complicated theories involving nucleon recoil, and nucleon-antinucleon intermediate states, do not succeed in explaining the experimental data, in particular the difference in sign between the scattering lengths a_1 and a_3. There is, however, a dispersion relation relating a_3, $-a_1$, f^2 and σ_{tot} which appears to be satisfied within the experimental errors.

Thus our treatment of s-wave phenomena will be largely phenomenological. The same applies to meson production in nucleon-nucleon collisions, though the favoured production of p-state pions at moderate energies is to be expected. At low energies, of course, s-wave production predominates.

The situation for photoproduced mesons is somewhat different. There is a theory which predicts the π^-, π^+ photoproduction cross sections and their ratio at threshold, knowing the coupling constant involved. However for a long time the experiments have tended to give answers different from the theory, though recent work by BALDIN and CINI *et al.* has brought better agreement. These remarks are included in order to indicate how the phenomenological scheme stands relative to the fundamental theory which we should have.

4) $\sigma \to \omega$. Link between positive energy processes outside the dotted box and zero energy processes, *i.e.* capture from K shell.

The connection can be understood from an elementary point of view, as pointed out by Fermi and Anderson.

Consider:

$$\underbrace{\pi^- + \text{p}}_{A} \to \underbrace{\pi^0 + \text{n} + Q}_{B} \text{ energy}.$$

We can pretend that the interactions involved are small perturbations, because at very low energies the cross sections are small compared with $\lambda\!\!\!^{-2}$. We may say that:

$$\omega_{AB} = \frac{2\pi}{\hbar} |H_{BA}|^2 \frac{\mathrm{d}N_B}{\mathrm{d}E}.$$

When the π^- is of very low energy, the phase space available to the π^0 and the neutron is determined entirely by the energy release Q. Hence, $\mathrm{d}N_B/\mathrm{d}E$ tends to a constant and we need only discuss the matrix element H_{BA}.

Thus we say:

$$\frac{\omega_s\ (\text{rate for low positive energy})}{\omega\ (\text{rate from } K\text{-shell})}=\frac{\left|\int\varphi_f^*(\boldsymbol{r})H_{\text{int}}\varphi_s(\boldsymbol{r})\,\mathrm{d}\tau\right|^2}{\left|\int\varphi_f^*(\boldsymbol{r})H_{\text{int}}\varphi_K(\boldsymbol{r})\,\mathrm{d}\tau\right|^2},$$

where $\varphi_s(\boldsymbol{r})$ is the initial wave function of the meson for low positive energy, $\varphi_K(\boldsymbol{r})$ is the wave function of the meson in the K shell, $\varphi_f(\boldsymbol{r})$ is the final wave function, and is independent of the incident meson energy for low energies, and H_{int} is some effective interaction.

Assume that the wavelength of the initial state is long compared to the range of interaction, and hence that the above ratio reduces to:

$$\frac{w_s}{w}=\frac{|\varphi_s(0)|^2}{|\varphi_K(0)|^2}.$$

(In the region of interaction

$$\varphi_s(\boldsymbol{r})\simeq\varphi_s(0)\,,\qquad \varphi_K(r)\simeq\varphi_K(0))\,.$$

We have

$$\varphi_s(\boldsymbol{r})=\exp[ikz]\,.$$

Hence

$$\lim_{k\to 0}\exp[ikz]=\frac{\sin kr}{kr}\simeq 1\,.$$

Therefore:

$$|\varphi_s(0)|^2=1\,.$$

Also:

$$\varphi_K(\boldsymbol{r})=\pi^{-\frac{1}{2}}b_0^{-\frac{3}{2}}\exp[-r/b_0]\,,$$

where b_0 is the Bohr radius. Hence:

$$\varphi_K(0)=\pi^{-\frac{1}{2}}b_0^{-\frac{3}{2}}$$

and

$$|\varphi_K(0)|^2=\frac{1}{\pi b_0^3}\,.$$

Therefore:

$$\frac{w_s}{w}=\frac{\text{incoming flux}\times\text{cross section}}{w}=\frac{v\sigma_s}{w}=\pi b_0^3\,,$$

$$\sigma_s=\frac{\pi b_0^3}{v}\,w=\left(\frac{\pi b_0^3}{c}\right)\frac{w}{\eta}\,,$$

where η is the momentum in units of μc.

As can be seen the cross section obeys the familiar $1/v$ law for low energy absorption.

5) r. Let us consider:

$$\gamma + \mathrm{p} \rightarrow \pi^+ + \mathrm{n} - Q\,.$$

We require

$$\sigma_s(\gamma,\, \mathrm{p};\, \pi^+,\, \mathrm{n})\,.$$

We know that $\sigma_s(\pi^+, \mathrm{n}; \gamma, \mathrm{p})$ obeys the $1/v$ law being an absorptive and exothermic process. Hence from detailed balance we say that:

$$\sigma_s(\gamma,\, \mathrm{p};\, \pi^+,\, \mathrm{n}) \propto \frac{1}{\eta}\,\frac{P_\pi^2}{P_\gamma^2}\,,$$

where P_π is the momentum of the pion, and P_γ that of the photon. P_γ is effectively constant near threshold, and hence:

$$\sigma_s(\gamma,\, \mathrm{p};\, \pi^+,\, \mathrm{n}) \propto \frac{1}{\eta}\cdot\frac{\eta^2}{\text{const}} \propto \eta\,.$$

The π^- to π^+ ratio is:

$$r = \frac{\sigma_s(\gamma,\, \mathrm{n};\, \pi^-,\, \mathrm{p})}{\sigma_s(\gamma,\, \mathrm{p};\, \pi^+,\, \mathrm{n})} \approx \frac{\sigma_s(\gamma,\, \mathrm{d};\, \pi^-,\, 2\mathrm{p})}{\sigma_s(\gamma,\, \mathrm{d};\, \pi^+,\, 2\mathrm{n})}\,,$$

if Coulomb interactions are neglected. From the value r_0 of r at zero energy we can calculate $\sigma_s(\gamma + \mathrm{n} \rightarrow \pi^- + \mathrm{p})$.

We could measure $\pi^- + \mathrm{p} \rightarrow \gamma + \mathrm{n}$, and obtain $\gamma + \mathrm{n} \rightarrow \pi^- + \mathrm{p}$ by detailed balance. However the cross section is very small compared to the charge exchange cross section, and this is a difficult experiment.

What should r_0 be theoretically? We have to consider what type of photoproduction takes place at low energy. Consider a nucleon absorbing a photon, and going into some fictitious state from which it emits a pion returning to a nucleon.

For the final nucleon:

$j = \frac{1}{2}\,, \qquad j_z = \mp\frac{1}{2}\,, \qquad \Pi$ (parity) $= +\,.$

For the pion at low energies:

$j = 0\,, \qquad j_z = 0\,, \qquad \Pi = -\,.$

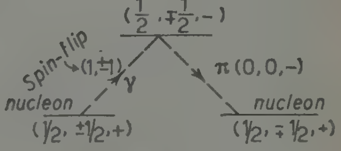

Fig. 3.

Thus, for the fictitious state:

$$j = \tfrac{1}{2}\,, \qquad j_z = \mp \tfrac{1}{2}\,, \qquad \Pi = - .$$

The initial nucleon has $j = \frac{1}{2}$, thus the transition to the fictitious state must be a dipole transition. If it were a quadrupole transition, or higher, then the photon would carry two units or more of angular momentum, and this would be incompatible with a transition from $j = \frac{1}{2}$ to $j = \frac{1}{2}$.

There is a change in parity, and hence the transition is electric dipole at threshold.

We know that for the (π^-, p) system the electric dipole moment is proportional to

$$-e\left(1 + \frac{\mu}{M}\right).$$

This can be seen from the diagram below (on the right):

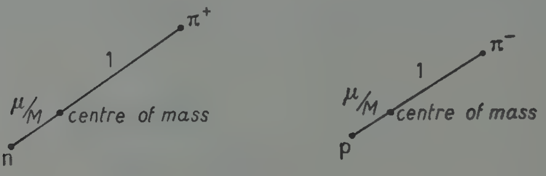

Fig. 4.

For the (π^+, n) system, the electric dipole moment is proportional to e as can be seen from the diagram on the left. The transition probability might be expected to be proportional to the square of the electric dipole moment.

Therefore:

$$r_0 = \frac{\sigma_s^-}{\sigma_s^+} = \frac{[-e(1+\mu/M)]^2}{e^2} \simeq 1.3\,.$$

These considerations, incidentally, explain why the reaction:

$$\gamma + p \to \pi^0 + p\,,$$

has such a small cross section at threshold. The electric dipole moment is proportional to (μ/M), and hence the cross section is only 2% of that for the above reactions.

6) S and R have both been measured experimentally; the link T is entirely theoretical, and is based on the argument that the proton in the deuteron undergoes the same interaction as a free proton.

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How does the interaction which causes the dipole transition arise? The usual interaction between $\boldsymbol{\sigma}_{\text{nucleon}}$ and $\boldsymbol{p}_{\text{nucleon}}$ is of the form $\boldsymbol{\sigma}\cdot\boldsymbol{p}$. In an electromagnetic field we replace $\boldsymbol{p}$ by $\boldsymbol{p}-(e/c)A\boldsymbol{\epsilon}$; $A\boldsymbol{\epsilon}$ is the vector potential, and $\boldsymbol{\epsilon}$ is the polarization vector. The interaction now becomes:

$$\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{e}{c}A\boldsymbol{\epsilon}\right).$$

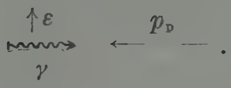
This gives an extra term acting on the nucleon:

$$\lambda(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}),$$

where λ is a constant.

Consider again the reaction:

$$\gamma+\mathrm{p}\rightarrow\pi^{+}+\mathrm{n}$$



Let the γ be incident along the z-axis: $(\varepsilon_z=0)$. We know that:

$$H_{\text{int}}=\lambda(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon})=\lambda(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y)=\tfrac{1}{2}\lambda[(\sigma_x+i\sigma_y)(\varepsilon_x-i\varepsilon_y)+(\sigma_x-i\sigma_y)(\varepsilon_x+i\varepsilon_y)].$$

$\sigma_x+i\sigma_y$ changes the z components of the nucleon spin by $+1$, whilst $\varepsilon_x-i\varepsilon_y$ absorbs a photon of spin $+1$, *i.e.* a right circularly polarized photon. Similarly $\sigma_x-i\sigma_y$ changes the z components by -1 unit, whilst $\varepsilon_x+i\varepsilon_y$ absorbs a left circularly polarized photon. Thus spin-flip terms arise from the replacement of $\boldsymbol{p}$ by $\boldsymbol{p}-(e/c)A\boldsymbol{\epsilon}$, as required from gauge invariance.

We can attempt to calculate T using the operator $\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}$ for H_{int}. Using the usual expression for transition rates, we obtain:

$$T=\frac{|\int\psi_{2\mathrm{n}}^{*}\psi_{\mathrm{d}}\,\mathrm{d}\tau\,(\chi_{2\mathrm{n}}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\chi_{\mathrm{d}})|^{2}}{|\int\psi_{\mathrm{n}}^{*}\psi_{\mathrm{p}}\,\mathrm{d}\tau\,(\chi_{\mathrm{n}}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\chi_{\mathrm{p}})|^{2}}.$$

But:

$$\int\psi_{\mathrm{n}}^{*}\psi_{\mathrm{p}}\,\mathrm{d}\tau=1.$$

Hence:

$$T=\frac{|\int\psi_{2\mathrm{n}}^{*}\psi_{\mathrm{d}}\,\mathrm{d}\tau\,(\chi_{2\mathrm{n}}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\chi_{\mathrm{d}})|^{2}}{|(\chi_{\mathrm{n}}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\chi_{\mathrm{p}})|^{2}}.$$

From now on, take the z axis along $\boldsymbol{\epsilon}$, to give:

$$\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}=\sigma_z.$$

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We get then:

$$\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\chi_d = \sigma_z\chi_d\,.$$

χ_d has three components which may be written symbolically as:

$$\frac{1}{\sqrt{3}}\begin{pmatrix} \mathrm{p}\;\mathrm{n} \\ \uparrow\;\uparrow \end{pmatrix},$$

$$\frac{1}{\sqrt{6}}\left(\begin{matrix} \mathrm{p}\;\mathrm{n} \\ \downarrow\;\uparrow \end{matrix} + \begin{matrix} \mathrm{p}\;\mathrm{n} \\ \uparrow\;\downarrow \end{matrix}\right),$$

$$\frac{1}{\sqrt{3}}\begin{pmatrix} \mathrm{p}\;\mathrm{n} \\ \downarrow\;\downarrow \end{pmatrix}.$$

When σ_z operates on the proton spin, these components become:

$$\frac{1}{\sqrt{3}}(\uparrow\uparrow),$$

$$\frac{1}{\sqrt{6}}(-\downarrow\uparrow+\uparrow\downarrow),$$

$$-\frac{1}{\sqrt{3}}(\downarrow\downarrow).$$

If the two neutrons are produced in an s-state, then by the Pauli principle it must be a singlet s-state, and hence:

$$\chi_{2n} = \frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow).$$

It is clear that $(\chi_{2n}, \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\chi_d)$ vanishes except when the deuteron is in the middle state $(m=0)$. In that case,

$$|(\chi_{2n}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\chi_d)|^2 = \left|\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow),\quad \frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)\right)\right|^2 = \frac{1}{3}.$$

We now require:

$$\left|\int \psi_{2n}^{*}\psi_d\,d\tau\right|^2.$$

Because two identical neutrons are involved

$$\int_{\text{half space}} \psi_{2n}^{*}\psi_{2n} = 1\,.$$

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Also ψ_{2n} and ψ_d overlap almost exactly, because the shape of the nearly bound ${}^1s$ wave function is nearly the same as the shape of the deuteron wave function:

$$\int \psi_{2n}^* \psi_d \, d\tau = \sqrt{2}\,.$$

Thus:

$$\left| \int \psi_{2n}^* \psi_d \, d\tau \right|^2 = 2\,.$$

Hence:

$$T = \frac{2}{3} \cdot \left(\frac{b_0^H}{b_0^D} \right)^3,$$

where the factor $(b_0^H/b_0^D)^3$ allows for the different Bohr radii in hydrogen and deuterium, brought about by the change in reduced mass of the pion.

The result for T was quoted by Brueckner *et al.*, and the simple derivation given here is due to Dalitz.

3. – Experimental results.

In the last two lectures, we have not considered the actual experimental results involved. We will now discuss these results starting with those for the zero energy ratios:

$$T = \frac{w(\pi^+ + d \to 2n + \gamma)}{w(\pi^- + p \to n + \gamma)}\,, \qquad R = \frac{w(\pi^- + p \to \pi^0 + n)}{w(\pi^- + p \to \gamma + n)}\,, \qquad S = \frac{w(\pi^- + d \to 2n)}{w(\pi^- + d \to 2n + \gamma)}\,.$$

We have already discussed T, and obtained a purely theoretical estimate. R was first measured, as is well known, by PANOFSKY *et al.* who obtained a value 0.94 ± 0.30. The γ-ray from the process $\pi^- + p \to \gamma + n$ is mono-energetic, having energy 130 MeV, whereas the γ-rays from π^0 decay have a distribution of energies centred on 70 MeV, due to the Döppler shift produced by the motion of the π^0 (Fig. 5).

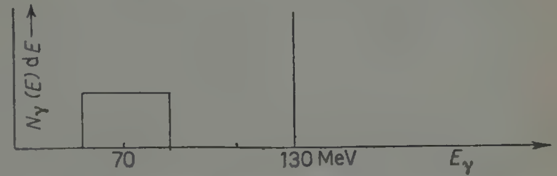

Fig. 5. – Energy distribution of γ-rays from π^- stopping in Hydrogen.

The experiment was carried out using a pair spectrometer to determine the energy-distribution of γ-rays from π^- capture in hydrogen. The γ-rays were made to produce electron-positron pairs in a lead radiator, and these pairs were counted in associated sets of counters arrayed as shown (Fig. 6) in a strong magnetic field.

The energy resolution of this instrument is good — it is easily able to resolve the γ-rays from the two processes, but the detection efficiency is low (~ 10^{-4} electron pairs/γ-ray) accounting for the rather large statistical errors assigned to this measurement.

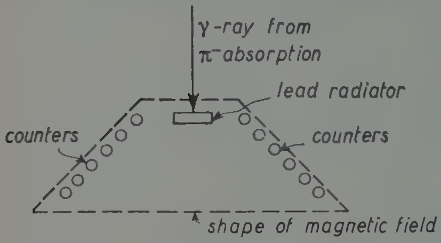

Fig. 6.

This result stood for five or six years, when further determinations were carried out. Two measurements were made at Liverpool, one, using a pair spectrometer, by KUENHER *et al.* who obtained a value 1.60 ± 0.17. The other was made simultaneously by CASSELS *et. al.* who used a total absorption spectrometer and obtained the value 1.50 ± 0.15.

In the total absorption spectrometer, the γ-ray is allowed to pass into a large volume of a heavy transparent medium where shower production occurs. All the energy is lost as ionization energy of the electron tracks. Now, nearly the whole track is relativistic and so the energy loss per unit length is constant, so the track length is proportional to the energy of the incident γ-ray. Further, the intensity of the Čerenkov radiation is a function only of v/c and the Čerenkov light per unit length of track is constant. Thus the total Čerenkov light is proportional to the energy of the incident γ-ray. However, since the intensity of the Čerenkov radiation is between 0.5 and 1.0 photoelectron per MeV of energy loss in the medium, the instrument has poor resolution. The efficiency is high, even if an external converter is used.

The distribution obtained was of the form of Fig. 7.

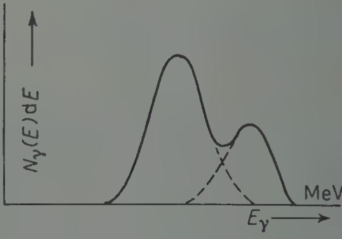

Fig. 7.

More recently, FISCHER and MARSHALL, also using a total absorption spectrometer, obtained the result 1.87 ± 0.10. Several experiments which are now in progress, should give more accurate results. Accepting the errors of the above methods and combining the results, one obtains a value:

$$R = 1.74 \pm 0.08 .$$

The value of S is measured less directly. We can detect the γ-ray in the process $\pi^- + d \rightarrow 2n + \gamma$. The two neutrons are produced in the $^1s_0$ state and so are almost bound; the result is almost mono-energetic γ-rays of the maximum energy. Incidentally, this γ-ray spectrum forms one of the better ways of studying the almost bound state of the two neutrons.

In the reaction $\pi^- + d \to 2n$, the neutrons are not detected, of course, by the γ-ray detectors and so an alternative way of evaluating the ratio is required. This is done by measuring the Panofsky ratio, and then, under the same experimental conditions, replacing the liquid hydrogen by liquid deuterium; it is assumed that the same number of pions stop in each, so that the total number of captures is the same for the two targets. Then, by comparing the γ-ray intensity for deuterium with the total for hydrogen, the fraction of the total number of pions stopping in deuterium which follow $\pi^- + d \to 2n + \gamma$ is obtained; hence by subtraction and inference, the fraction which follow $\pi^- + d \to 2n$ is obtained. This was done in Panofsky's original experiment and resulted in a value 2.36 ± 0.74.

We will see later on that the fact that the second of the above reactions occurs is the strongest argument for saying that the pion is pseudoscalar. In view of this, a more direct verification that it does take place was obtained by Chinowsky and Steinberger who used counters to detect the two neutrons. They obtained a value $S = 1.5 \pm 0.8$.

Kuehener *et al.* repeated the Panofsky experiment with an improved pion source and obtained a value 2.36 ± 0.36. Combining the above results, we obtain an average value

$$S = 2.31 \pm 0.31\,.$$

The values of T, R and S complete our knowledge of the zero energy processes.

4. – Extrapolation to zero energy.

We are now going to approach the zero energy box by the three positive energy chains which start at

(a) $\sigma_s(p + p \to \pi^+ + d)\,.$

(b) $\sigma_s(\gamma + p \to \pi^+ + n)\,,$

(c) $\sigma_s(\pi^+ + p \to \pi^0 + n)\,.$

The best measurement of cross-section *(a)* was carried out by Crawford at Berkeley, who obtained $\sigma_s = (0.125 \pm 0.017)\eta$ mb. The experiment is performed by counting coincidences between π^+ and d which prevents the detection of protons and neutrons produced in the unbound state.

Collecting together the experimental results, we are now in a position to write down the capture rates tabulated below.

Process	10^{14} w (s^{-1})	Input data	Comments
$\pi^- + p \to \pi^0 + n$	5.67	$a_1 - a_3 = 0.24$	Comes from scattering data. Value originally 0.27, but was revised by CINI.
		$v_0/c = 0.20$	Velocity of π^0 at low energy, which enters into the $1/v$ law.
$\pi^+ + p \to \gamma + n$	3.69	$\sigma_s(\gamma + p \to \pi^+ + n)$ $= 1.87 \cdot 10^{-28}$ cm²	Comes from photoproduction. BERNARDINI' svalue was $1.43 \cdot 10^{-28}$ cm². Revised by CINI.
		$r_0 = 1.3$	This is the theoretical value discussed earlier.
$\pi^- + d \to 2n$	6.02	$\sigma_s(p + p \to \pi^+ + d)$ $= 1.25 \cdot 10^{-28}$ cm²	Obtained from the experiment of CRAWFORD *et al.*

Let me emphasize that no zero energy experiments are involved in the derivation of the above reaction rates. They are all positive energy experiments.

We can now take ratios of these three rates and compare with the ratios obtained by zero energy experiments. This will provide a consistency test of the theory, which is essentially phenomenological, except for the photoproduction process.

5. – Comparison of rates deduced from positive energy experiments with the *K*-shell capture rates.

We use the revised results of CINI *et al.*

Ratio	Positive energy rate	*K*-shell rate
R	1.53	1.74
ST	1.63	1.88
ST/R	1.06	1.08

Note: $ST = \dfrac{w(\pi^- + d \to 2n)}{w(\pi^- + p \to n + \gamma)}$, $\quad \dfrac{ST}{R} = \dfrac{w(\pi^- + d \to 2n)}{w(\pi^- + p \to \pi^0 + n)}$.

For many years the agreement between these values was poor, but we now arrive at a much more satisfying situation where the agreement looks quite good. It is gratifying to note that about two years ago, there was reasonable consistency in these numbers, but it was achieved only by having disturbing values for the photoproduction process which were not too easily acceptable. The trouble was that it was difficult to see how r_0 could be any value other than about 1.3, and further, the value for the π^+ photo-production cross-section was too low. It was possible to make a reasonable fundamental theory for this process at threshold, and this low value of the production cross-section led to a value of f^2 which was considerably smaller (about 0.05 or 0.06) than was indicated by other routes of enquiry.

So, by lowering r_0 and raising the cross-section, the experiments have now been brought into rather good line with this fundamental theory of photoproduction. The big steps which led to this were the re-analysis of the π^+ cross section by Cini *et al.* and the discussion of the value of R by Baldin who showed that the deuterium data were consistent with the correct value of r_0.

When the proper value of r_0 was first used to obtain a positive energy value of the Panofsky ratio, none of the other input data were altered and the resultant value of R was about 2.5.

This led BALDIN to suggest the possible existence of a pseudoscalar neutral meson of isotopic spin zero (the π^0_0), produced in the absorption of π^- by protons, which also decays into two γ-rays.

There are two other reasons to support the invention of the π^0_0:

a) there appears to be a vacancy for such a particle in the general scheme of fundamental particles;

b) experiments on nucleon-nucleon production of pions at Birmingham appear to violate charge independence, but could be explained by the existence of a π^0_0-meson.

6. – Proof that the pion is pseudoscalar.

This is shown by the reaction $\pi^- + d \rightarrow 2n$. Since the pion has zero spin and the deuteron has spin 1, and since the (π^-, d) system is in an s state, the system has total angular momentum $J=1$. Consequently, the 2n-state has $J=1$ and so has odd parity. Then, since the deuteron has even parity, the pion must have odd parity.

A suggestion made some years ago, by YANG, for investigating the parity of the neutral pion involved the determination of the relative planes of polarization of the two γ-rays into which the π^0 decays. The method was to allow the γ-rays to produce positron-electron pairs, when the plane defined by the e^+ and e^- momentum vectors defines the plane of polarization of the γ-rays. However, the angle between the electron pairs is small, and the material introduced to cause pair production by the γ-rays also causes multiple scattering of the electrons so that the definition of the electron-positron plane is destroyed.

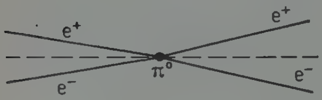

Fig. 8.

Dalitz first pointed out that in about 1/80 of the π^0 decays, the γ-rays are replaced by internally converted electron-positron pairs. So in about 1 in 10^4 of the decays, both γ-rays are internally converted, resulting in a « Pion Bow Tie » (Fig. 8).

Thus, we have conversion without the aid of a scatterer.

Some groups, notably those of Steinberger and Hildebrand are taking very many bubble chamber photographs and are selecting « Bow Tie » events, from which a determination of the π^0 parity will be made.

7. – π^0 mass.

The π^-, π^0 mass difference also appears from the Panofsky experiment. In the center of the mass system of the π^0, the two γ-rays are emitted at 180° to each other. Now, since there is an energy release in the K-shell reaction

$$\pi^- + p \to \pi^0 + n + Q\,,$$

the π^0 will have a definite energy in the lab. system dependent on the π^--π^0 mass difference and the p-n mass difference. Thus, in the laboratory system the γ-rays have a distribution in angle and in energy. The maximum and minimum γ-ray energies thus tell the magnitude of the center of mass velocity and consequently gives the π^--π^0 mass difference, the n-p mass difference being known. It may easily be shown that the energy distribution of the γ-rays is rectangular. The π^0_0 would add a further rectangular distribution as shown in Fig. 9.

However, the experimental errors are so large that one cannot say whether this modification exists or not.

A more powerful way is to look at the angular correlation of the two γ's which are no longer at 180° in the laboratory, because of the π^0 velocity. The angular distribution has the form of Fig. 10.

This rather encouraging distribution was used by Chinowsky and Steinberger to measure the π^0 mass, but again, the accuracy is such that the addition of another distribution of a similar shape, inside this one corresponding to a slightly lower π^0-π^- mass difference, could not be completely excluded. Thus, this experiment is worth repeating more accurately, when it would also give a more accurate mass difference. At the moment, it is known to about 300 keV.

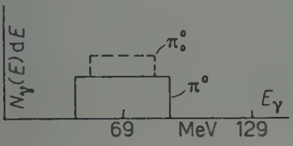

Fig. 9.

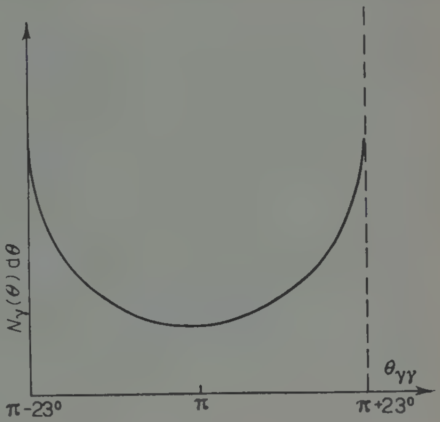

Fig. 10.

8. – Mesonic X-rays from more complicated nuclei than H and D.

Consider a nucleus with Z protons and N neutrons, and a π^- in orbit round this nucleus in the K-shell. Consider also the scattering by this nucleus of low energy π^- which might come from outside.

The scattering cross section at low energy is obtained by adding coherently the scattering lengths of all the Z protons and N neutrons:

$$\sigma = 4\pi\bar{\lambda}_c^2[(N+\tfrac{1}{3}Z)a_3+\tfrac{2}{3}Za_1]^2 .$$

We do not know the interaction which is responsible for the S-wave pion, nucleon scattering, so it is necessary to invent one in the usual way. Thus:

$$\sigma = \frac{2\pi}{\hbar v}\left|\int \exp[i(\boldsymbol{k}_i-\boldsymbol{k}_f)\cdot\boldsymbol{r}]\,V(r)\,\mathrm{d}\tau\right|^2\frac{\mathrm{d}N}{\mathrm{d}E},$$

where: $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$ are the initial and final pion wave vectors, $V(r)$ is the assumed interaction potential and $\mathrm{d}N/\mathrm{d}E$ is the energy density of final states. Now, at very low non relativistic energies, k becomes very small and we may replace the exponential factor by 1, and evaluate $\mathrm{d}N/\mathrm{d}E = \mu^2 v/2\pi^2 h^3$ at low pion energy. We then get, taking square roots:

$$\int V\,\mathrm{d}\tau = -2\pi\frac{\hbar^2}{\mu}\bar{\lambda}_c\left\{\left(N+\frac{1}{3}Z\right)a_3+\frac{2}{3}Za_1\right\}.$$

The sign is correct because the positive scattering lengths correspond to an attractive potential, which is described by a negative V.

Now we will write down the expression for the L to K-shell meson transition energy assuming a pure electromagnetic interaction and obtain an expression for the energy shift due to perturbation by the nuclear potential. This finally gives us an expression for $\Delta E/E$ in terms of the S-wave scattering length s.

The transition energy for the pure electromagnetic interaction is:

$$E = E_K - E_L = \left(-\frac{\hbar^2 Z^2}{2\mu b_0^2}\right) - \left(-\frac{\hbar^2 Z^2}{2\mu b_0^2}\cdot\frac{1}{4}\right) = -\frac{3\hbar^2 Z^2}{8\mu b_0^2}\,,$$

where $E_{K,L}$ is the energy in the K or L-shell, and b_0 is the Bohr radius of an electron in the K-shell.

The shift in the K-shell level may be obtained by the perturbation metho and is:

$$\delta E = \int \varphi_K^* V \varphi_K \,\mathrm{d}\tau\,.$$

There is no similar term for the L-shell as the L-shell meson wave function is zero at the nucleus.

We assume that the K-shell meson wave function is constant over the nuclear volume and so we may write:

$$\delta E = |\varphi_K(0)|^2 \int V \,\mathrm{d}\tau\,.$$

This integral has already been obtained and the value of $|\varphi_K(0)|^2$ which we have already used in connection with the Panofsky ratio gives:

$$\delta E = \frac{Z^3}{\pi b_0^2} \int V \,\mathrm{d}\tau\,.$$

Thus the expression for the level shift is:

$$\frac{\Delta E}{E} = \frac{16}{3} Z \frac{\mu}{\mu_H} \frac{\bar{\lambda}_c}{b_0} \left\{\left(N + \frac{1}{3} Z\right) a_3 + \frac{2}{3} Z a_1\right\} \simeq \text{const}\cdot(2a_3 + a_1) \simeq -0.06$$

from scattering experiments.

The quantity μ_H is the reduced mass of the pion-nucleon system and originates in the calculation of $\int V \mathrm{d}\tau$ from pion-nucleon scattering data, as the number of nucleons increases this reduced mass approaches the free meson mass μ.

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In addition to the above considerations there is the added complication of reactions of the type:

$$(a) \qquad \pi^{-}+\mathrm{p}+\mathrm{p}\rightarrow\mathrm{p}+\mathrm{n}\,,$$

and these add other terms to the level shift.

The experimental value of the level shift is negative and is rather greater than the value given by the Orear values of the scattering lengths.

This increase is attributed to the reaction (a). The calculation of the correction is however very sensitive to the nucleon momentum distribution in the nucleus as this determines how easily the meson can get into the strongly interaction P state with respect to the two protons.

Any calculation is therefore rather sensitive, but some agreement with experiment has been achieved.

Essentially, there have been three measurements at:

1) Rochester by CAMAC *et al.*;
2) Carnegie Tech. by STEARNS and STEARNS;
3) Liverpool by WEST (A.E.R.E. Harwell).

The first two used the critical absorption method in which absorbers are placed between the X-ray source and a NaI counter. Various elements are used in turn until a large attenuation of the X-rays is obtained. This then allows the energy to be determined between the limits of two known K-absorption edges.

West's technique employed a Xenon proportional counter in which he measured the X-ray energies against calibration lines. The width of the Be K-line was also measured and found to be approximately 1 keV. This was done by comparing the line to the Fluorine L-line where the capture rate is low, and hence the line narrow.

The final stage of this sort of measurement will probably be the measuring of these widths, which is just beginning to be possible, and thus obtain the life-time of the π^{+}-meson in the K-shell. This in turn will give further information on the nuclear absorption processes. The value of $(2a_3+a_1)$ will probably be determined more accurately by scattering experiments in the future. So the end product of this sort of measurement will be an accurate determination of the nucleon momentum distribution which, as has been pointed out, is very sensitive to the level shift.

STEARNS and STEARNS have made an accurate measurement of the energy of the 4F to 3D transition in phosphorous. This line will not be perturbed by the nuclear potential as both wave functions are small at the nucleus and so this gives an accurate determination of the π^{-} mass.

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N. 2, 1959
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Elementary Considerations on Photo Meson Production.

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There is a strong connection between the photo-mesic effect and the scattering of mesons by nucleons. I will illustrate this by showing that the phases for photoproduction are equal (mod π) to the scattering phases. The argument is due to WATSON and can be compressed thus:

Let $S=I+iR$ be the S-matrix and let us confine our attention to that range of energy, where double meson production is not yet possible. It follows from the unitarity of S that

$$R-R^{+}=iR^{+}R\,. \tag{1}$$

We apply this equation to the state of photoproduction. Denoting by $|\gamma)$ a state with a nucleon and a photon present and by $(\pi|$ a state of the same energy with a nucleon and a meson we have

$$(\pi|R|\gamma)-(\gamma|R|\pi)^{*}=i\sum_{n}(\pi|R^{+}|n)(n|R|\gamma)\,, \tag{2}$$

where the sum is extended over all the states which may be reached from $|\gamma)$ under conservation of energy. By hypothesis there are only two classes of states available, namely $|n)=|\pi)$ and $|n)=|\gamma)$. The contribution of $|n)=|\gamma)$ however is small of order $e^{2}=1/137$ compared with the contribution of $|n)=|\pi)$. (Remember that $(\gamma|R|\gamma)$ must be of order e^{2} and that $(\pi|R|\gamma)$ is of order e.)

With an expected accuracy of about one percent we can therefore put

$$(\pi|R|\gamma)-(\gamma|R|\pi)^{*}=i\sum_{\pi'}(\pi'|R|\pi)^{*}\,(\pi'|R|\gamma)\,. \tag{3}$$

If we use for the pions a representation in which R is diagonal the sum in (3) reduces to a single term. We can then express the diagonal elements of R by the phases:

$$(\pi|R|\pi) = 2e^{i\delta}\sin\delta\,. \tag{4}$$

In order to determine the photoproduction phases we make use of reversibility. In a time reversible theory we must have

$$(\pi|R|\gamma) = \eta(\gamma|R|\pi)\,, \tag{5}$$

where η can only assume the values ± 1. η can be determined to be $(-)^l$, where l is the angular moment of the pions. To see this take for example s-wave production, for which we may write

$$(\pi|R_s|\gamma) = \alpha\varphi(\boldsymbol{e}\boldsymbol{\sigma})\,. \tag{6}$$

Here α is a c-number, $\boldsymbol{e}$ is the vector of polarization of the photon σ, is the nucleon spin operator and φ is the field operator of the meson. Since under time reversal σ and φ change sign (the first because it is an angular momentum, the second because the meson field operator is a pseudoscalar), we have $\eta = 1$ in this case.

Putting now $(\pi|R|\gamma) = re^{ix}$, with r = real and positive and inserting into (3) we obtain

$$\left\{\begin{array}{ll} x = \delta (\mathrm{mod}\ \pi) & \text{for}\quad \eta = +1 \\ x = \frac{1}{2}\pi + \delta\,(\mathrm{mod}\ \pi) & \text{for}\quad \eta = -1 \end{array}\right. \tag{7}$$

and we therefore see that reversibility and unitarity of the S-matrix together with the neglect of the « Compton phases » establish the equality mod π of scattering and photoproduction phases.

One could on the basis of the knowledge of the scattering phases try to establish a phenomenological theory of photo meson production. However, it turns out that for the 4 processes

$$\left\{\begin{array}{l} \gamma + \mathrm{p} \begin{array}{l} \nearrow \mathrm{n} + \pi^+ \\ \searrow \mathrm{p} + \pi^0 \end{array} \\ \gamma + \mathrm{n} \begin{array}{l} \nearrow \mathrm{n} + \pi^0 \\ \searrow \mathrm{p} + \pi^- \end{array} \end{array}\right. \tag{8}$$

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a phenomenological description still involves 12 unknown functions, even if one limits the discussion to S and P waves. The situation is therefore considerably worse than in the theory of scattering. Moreover it is nearly certain that an expansion in partial waves will not be very satisfactory, since retardation effects of the electromagnetic field make such an expansion converge very slowly.

On the other hand we have seen in the discussion of the fixed source model that apart from the 33-state there is no reason to disbelieve perturbation theory at low energies. The coupling constant turned out to be fairly small (of order 0.1), so that one would expect that all that remains to be done for an adequate description is to apply a final state correction for the 33-state. This indeed gives reasonable agreement with experiment.

I will now report on some work carried out by STOPPINI and myself in 1956. We have not yet tried to bring the comparison up to date, but it appears that experimental data are in better agreement with the simple minded theory which I am now going to present, than they were in 1956.

The basic idea is the recognition of the necessity to take account of both final state correction (in the 33-state) and anomalous magnetic moment interaction in order to obtain an adequate description of photomesic phenomena. We assume that R_0 is the operator of photomeson production as determined by means of perturbation theory and including the anomalous magnetic moment term. (The anomalous magnetic moment is defined by putting the magnetic moment μ of the nucleon $=\mu_1+\mu_2\tau_3$; μ_2 is the the anomalous magnetic moment in the sense in which we are going to use it.) We consider only the lowest terms in an expansion in powers of $1/m$. We then maintain that instead of R_0 one should compare

$$R = R_0 + (S-1)P_{33}R_0 \tag{9}$$

with experiment. In (1) P_{33} is the projection operator which, when applied to R_0, filters out the P_{33}-component and S (in the representation of good variables) is a complex number, which corrects for the deformation of the meson wave near the nucleus. Assuming short range forces of range r_0, $S=\lim_{r\to 0}\psi^*(r)/\psi_0^*(r)$ (where $\psi(r)$ is the wave function of a physical meson, $\psi_0(r)$ its Born-approximation both normalized in such a way that their outgoing parts coincide), is estimated to be

$$S = \frac{3}{(pr_0)^3}\exp[i\alpha_{33}]\sin\alpha_{33}\,. \tag{10}$$

Here α_{33} is the scattering phase and p is the meson momentum in rational units ($\hbar=c=1$).

Equation (9) gives for the scattering amplitude

$$R = \frac{\alpha}{k}\Big\{(\tau_1 a_2^+ - \tau_2 a_1^+)\left(\sigma_1 + \frac{\sigma_3 - (\sigma' v)}{1 - v_3} + ixv_2\right) + ixa_3^+(\sigma_3 v_1 - \sigma_1 v_3) + \\ + \frac{1}{g}(S-1)(\tau_1 a_2^+ - \tau_2 a_1^+ + 2ia_3^+)(D\sigma_3 v_1 + F\sigma_1 v_3 + iGv_2)\Big\}, \tag{11}$$

where α is a constant, k is the momentum of the incident photon, τ_i are the isotopic spin operators, a_i^+ and a_i the creation and destruction operators for the mesons ($i = 3$ describing a neutral meson, $i = 1, 2$ charged particles), v is the velocity of the emergent meson in the zero momentum system, the photon has been assumed to move along the z-axis and to be polarized along the x-axis, x is defined as $\mu_2 k/m$, and D, F and G are defined by

$$D = 2 - C + 2x, \tag{12}$$

$$F = 1 - 2C - 2x, \qquad G = 1 + C + 4x.$$

$C = C(v)$ is given by

$$C(v) = \frac{3}{4} v\left\{\frac{2}{v} - \frac{4}{3} v - \frac{1-v}{v^2}\log\frac{1+v}{1-v}\right\}. \tag{13}$$

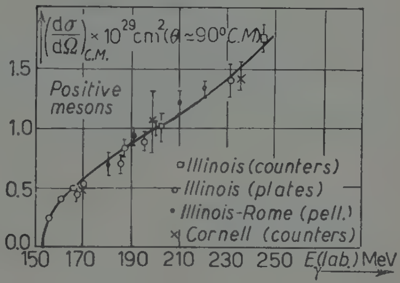

Fig. 1.

It behaves like $v^2/5$ for small v and approaches the value $\frac{1}{2}$ in the extreme relativistic limit (very reluctantly). The first term in the curly bracket of (11) corresponds to the perturbation amplitude for the production of charged mesons, the second to the perturbation amplitude for neutral mesons; the last term is the scattering term.

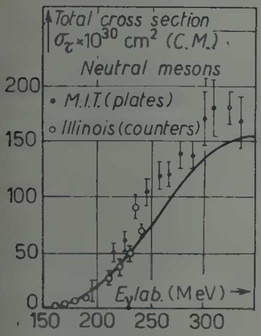

Fig. 2.

The main shortcomings of a perturbation theory which does not take account of the magnetic moment interaction and of final state corrections are the following:

1) The ratio π^0/π^+ comes out wrong, namely 0.

2) The angular distribution of the π^+ at moderately high energies is not in agreement with experimental evidence.

The introduction of the magnetic moment term makes one understand at least part of the π^0 intensity, but gives a wrong angular distribution for these mesons, namely $1+\cos^2\theta$, instead of the observed $\varepsilon+\sin^2\theta$ ($\varepsilon \ll 1$) suggested by experiment.

The introduction of the final state correction alone (*i.e.* without the Pauli term) gives a reasonable π^0/π^+ ratio namely $\frac{1}{2}$ in the limit $S \to \infty$, *i.e.* for a strong nucleon meson interaction. The total cross-sections for π^+ and π^0-production can be fitted satisfactorily, but the angular distribution for positive mesons comes out qualitatively wrong: as one can see from equation (11) the sign of the interference term is determined by F; for $x=0$, *i.e.* neglecting the anomalous magnetic moment effect, F varies between 1 at low energies to 0 at high energies. The interference term is therefore always positive in contrast to observation (there is a small anisotropy of the form $\sin^2\theta\cos^2\theta$ due to the meson current term, which has the right sign, but which does not by itself lead to agreement with experiment).

The inclusion of the magnetic moment term nearly solves this difficulty, as we can see from Figs. 3, 4, 5 representing the angular distributions at respectively 200, 235 and 265 MeV (the curves shown correspond to $r_0\mu = 0.66$ and $f^2 = 0.075$; in the evaluation of the cross-sections account has been taken of the kinematic effects of nucleon recoil). The agreement at 265 MeV is not very good, but this may be perhaps excused since at this energy recoil effects will almost certainly enter into play.

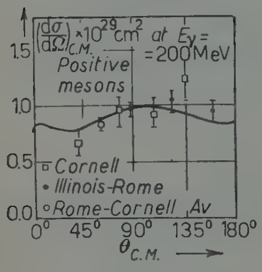

Fig. 3.

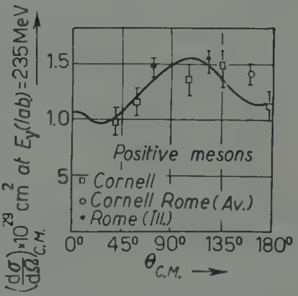

Fig. 4.

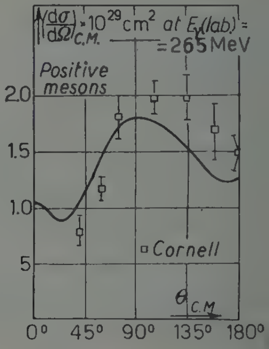

Fig. 5.

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SUPPLEMENTO AL VOLUME XIV, SERIE X
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N. 2, 1959
4º Trimestre

Meson Phenomena.

S. Fubini
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During the Course prof. Bernardini held three compendious lectures about *Photoproduction meson phenomena*, which, substancially, were conforming to part of those prof. S. Fubini held at CERN on the same subject, in spring 1958.

As Fubini's lectures are a wider treatement and a deeper illustration of the question, together with professors Bernardini and Fubini, we agree on reporting here Fubini's lectures in full.

Introduction.

In these lectures we shall examine some aspects of the theory of pion-nucleon interaction.

We shall base our treatment on a simplified version of the theory in which the nucleon is kept fixed and can emit and absorb mesons in p-wave state. Since it has been found experimentally that the low-energy pion phenomena are dominated by the p-wave (33) resonance, this simple model already gives an adequate description of the main features of phenomena involving pions and nucleons.

The static meson theory describes the interaction of the meson field with the spin and isotopic spin of the fixed nucleon.

We shall not treat in detail the quantization of the meson field, but for this we would refer, for example, to the Weisskopf lectures at CERN.

In the first Section we shall discuss the structure of the pion-nucleon interaction Hamiltonian. The second Section will be dedicated to the general description of the physical nucleon state.

We shall then discuss the predictions of the theory about pion-nucleon scattering (Section **3**) and photoproduction (Section **4**).

1. – The hamiltonian.

The meson field has three components: $\Phi_3(x)$ which represents the neutral mesons, and $\Phi_1(x)$, $\Phi_2(x)$ which together represent charged mesons.

The total energy of this meson field in the absence of interaction (the free Hamiltonian) is given by:

$$H_0 = \frac{1}{2}\int \sum_{1\alpha}^{3} [\pi_\alpha^2 + (\nabla\Phi_\alpha)^2 + \mu^2\Phi_\alpha^2]\, d^3x\,. \tag{1.1}$$

π_α is the momentum conjugated to Φ_α satisfying the relation:

$$[\Phi_\alpha(x), \pi_\alpha(x')] = i\delta_{\alpha\alpha'}\,\delta(x - x')\,. \tag{1.2}$$

The operator Q, representing the total charge of the meson field, is given by:

$$Q = +\, e\int [\Phi_1(x)\pi_2(x) - \Phi_2(x)\pi_1(x)]\, d^3x\,. \tag{1.3}$$

From (1.2) and (1.3) it follows that:

$$[(\Phi_1 + i\Phi_2), Q] = -\, e(\Phi_1 + i\Phi_2)\,, \tag{1.4a}$$

$$[(\Phi_1 - i\Phi_2), Q] = +\, e(\Phi_1 - i\Phi_2)\,, \tag{1.4b}$$

$$[\Phi_3, Q] = 0\,. \tag{1.4c}$$

Let us denote by $|Q'\rangle$ an eigenstate of the operator Q corresponding to the eigenvalue Q'; then we have:

$$Q|Q'\rangle = Q'|Q'\rangle\,.$$

If we now apply to $|Q'\rangle$ the first of eqs. (1.4) we get:

$$[(\Phi_1 + i\Phi_2), Q]|Q'\rangle = -\, e(\Phi_1 + i\Phi_2)|Q'\rangle$$

or

$$Q(\Phi_1 + i\Phi_2)|Q'\rangle = (Q' + e)(\Phi_1 + i\Phi_2)|Q'\rangle$$

which can be interpreted by saying that $(\Phi_1 + i\Phi_2)|Q'\rangle$ is an eigenstate which corresponds to the eigenvalue $Q' + e$, *i.e.* the operators $\Phi_1 \pm i\Phi_2$ applied to a state of charge Q' create states of charge $Q' \pm e$. Instead, according to (1.4c), Φ_3 does not change the charge of the meson field.

Recalling that Φ_1 can be expanded in terms of destruction and creation

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operators of mesons, we can conclude that:

$\Phi_1 - i\Phi_2$	is an operator which destroys a *positive meson* or creates a *negative meson*;
$\Phi_1 + i\Phi_2$	is an operator which creates a *positive meson* or destroys a *negative meson*;
Φ_3	contains creation and destruction operators of a neutral meson.

Let us now discuss the general assumptions we want to impose on the term H' in the Hamiltonian H, which represents the interaction of the meson field with the (fixed) nucleon.

We shall require that:

a) the nucleon source *is not allowed to move* and therefore its only possible states $\psi_{r,s}$ ($r = 1, 2$; $s = 1, 2$) are given by the orientation of the spin and isotopic spin. Therefore the only operators related to the source are the spin $\boldsymbol{\sigma}$ and the isotopic spin $\boldsymbol{\tau}$;

b) the theory is invariant under the rotation group and space inversion, *the meson field Φ_λ being pseudoscalar*;

c) the theory is charge-independent. This means that the total Hamiltonian must commute with the operators of the total isotopic spin defined (see Appendix I) as

$$T_\lambda = \tfrac{1}{2}\tau_\lambda + t_\lambda\,,$$

where

$$t_\lambda = \varepsilon_{\lambda\mu\nu}\int(\Phi_\mu\pi_\nu - \Phi_\nu\pi_\mu)\,\mathrm{d}^3x \quad (^*)\,.$$

d) The interaction Hamiltonian is linear in the meson field.

We see that requirement *a*), as already discussed, is a consequence of our static approximation; *b*) and *c*) are supported by very good experimental evidence. Finally, *d*) has been first suggested by YUKAWA in analogy with the interaction of an electron with the electromagnetic field. In our opinion, it is the weakest of these hypotheses, and it will be shown that many of the results of this paper do not depend critically on *d*).

(*) The symbol $\varepsilon_{\lambda\mu\nu}$ (where $\lambda, \mu, \nu = 1, 2, 3$) is defined to be $+1$ for $\lambda = 1, \mu = 2, \nu = 3$; and to be $+1$ or -1 for any other set of values of λ, μ, ν obtained from the former one by an even or an odd permutation. It is zero in the other cases.

The only interaction Hamiltonian satisfying the conditions *a*), *b*), *c*) and *d*) is:

$$H' = (4\pi)^{\frac{1}{2}}\frac{f_0}{\mu}\sum_\alpha\int U(x)\tau_\alpha\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\Phi_\alpha\,\mathrm{d}^3x\,. \tag{1.5}$$

The combination $\sum_\alpha \tau_\alpha\Phi_\alpha$ which appears in (1.5) is due to the charge-independence. In fact, $\boldsymbol{\tau}$ and $\boldsymbol{\Phi}$ are vectors in the space of isotopic spin; then their scalar product $\boldsymbol{\tau}\cdot\boldsymbol{\Phi}$ is invariant for rotations in such a space. One can verify such an invariance directly by observing that:

$$[T_\lambda, H'] = 0 \qquad \text{or} \qquad \sum_\alpha [T_\lambda, \tau_\alpha\Phi_\alpha] = 0\,.$$

In fact

$$\sum_\alpha [T_\lambda, \tau_\alpha\Phi_\alpha] = \sum_\alpha\{\tfrac{1}{2}[\tau_\lambda\tau_\alpha]\Phi_\alpha + [t_\lambda\Phi_\alpha]\tau_\alpha\} = \sum_{\alpha\mu}\{i\varepsilon_{\lambda\alpha\mu}\tau_\mu\Phi_\alpha + i\varepsilon_{\lambda\alpha\mu}\Phi_\mu\tau_\alpha\} = 0$$

where we used the relation $[t_\lambda, \Phi_\mu] = i\varepsilon_{\lambda\mu\nu}\Phi_\nu$ of Appendix I (A.I.5).

The term $\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\Phi$ has the following origin: as nucleon operators in the ordinary space we have only the identity operator 1 (scalar) and $\boldsymbol{\sigma}$ (pseudovector). If we want the theory to be linear in the meson field, we can only build up the following couplings, which are invariant for rotations in ordinary space:

$$\Phi \qquad \text{and} \qquad \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\Phi\,.$$

The first is pseudoscalar and therefore will not conserve the parity. Then the only term which can be included in the theory is $\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\Phi$.

f_0 is a constant of the theory called the unrenormalized coupling constant. The mass μ at the denominator is added in order to have f_0 as a dimensionless constant. It is required to compensate the extra (length)$^{-1}$ introduced by the operator $\boldsymbol{\nabla}$.

$U(x)$ is a spherically symmetric normalized function representing the size of the nucleon source. We will never be obliged to make too stringent assumptions about the form of $U(x)$. We shall only ask that its range R (*) be rather smaller than the Compton wave-length of the pion. This is to ensure that our static non-relativistic model has any chance of having something in common with the relativistic local theory in the approximation where terms of the order of μ/M (recoiling terms) are considered small.

Here we would like to remark on consequences arising from the fact that f_0/μ has the dimensions of 1/energy, *i.e.* on the consequence that the coupling depends upon the *gradient* of the field.

(*) R defines the region in which the function $U(x)$ is essentially different from zero.

This makes the results of the theory strongly dependent on the cut-off. For example, let us consider the self-energy of the nucleon, *i.e.* the change in the energy of the nucleon due to the successive emission and absorption of one or many mesons. If we write the perturbative expansion of the self-energy, we have:

$$\varDelta E = \left(\frac{f_0}{\mu}\right)^2 \varepsilon_2 + \left(\frac{f_0}{\mu}\right)^4 \varepsilon^4 + \dots \; (^*) \, ,$$

where ε_2, ε_4, ... are integrals. It will be shown later (see Eq. (1.7)) that the functions to be integrated depend only on k, ω_k and v_k, and that the high-energy region where $\omega_k \simeq k$ is the most important one. Therefore the integrals depend essentially on $k_{\max}$. Because of the dimensions of f_0/μ, it follows that ε_2 behaves as $(k_{\max})^3$, ε_4 as $(k_{\max})^5$, etc. Therefore, all orders of infinities would appear if we let $k_{\max}$ go to infinite, that is R going to zero (point source) (**). Such a theory is said to be unrenormalizable.

For the sake of comparison, let us consider a scalar interaction:

$$(4\pi)^{\frac{1}{2}} g_0 \sum_{\alpha} \int \tau_\alpha \varPhi_\alpha U(x) \, \mathrm{d}^3 x \, .$$

The perturbative expansion of ΔE would be:

$$\Delta E = g_0^2 \varepsilon_2 + g_0^4 \varepsilon_4 + \dots \, .$$

In the same way as before, we will conclude that:

$$\varepsilon_2 \sim \varepsilon_4 \sim \dots \varepsilon_n \sim k_{\max}$$

and we see that only a linear infinity appears in the expansion of ΔE.

Such an unphysical infinity could be removed by a finite number of subtractions and the theory is said to be renormalizable.

An important property of the Hamiltonian (1.5) is that only the mesons in p-wave state can be emitted or absorbed by the (fixed) source.

This can be physically understood by considering that for conservation of angular momentum the fixed nucleon can emit the pion only in s- or p-waves.

(*) Only even powers of the coupling strength appear, because at each order two elementary processes are coupled: the emission and the absorption.

(**) We remember here that because $U(k) = \int U(x) \exp[ikx] \, \mathrm{d}^3 x$, from the theory of Fourier transforms it follows

$$k_{\max} \simeq 1/R \, .$$

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Since the pion is pseudoscalar, parity conservation requires that it is emitted in a p-wave state. It is therefore very convenient to rewrite our Hamiltonian in terms of creation and destruction of mesons of a given angular momentum. We introduce the eigenfunctions $f_{klm}(\boldsymbol{x})$ of momentum k and angular momentum l, m (with respect to the nucleon).

These eigenfunctions satisfy the wave equation:

$$(\Delta + k^2)\, f(\boldsymbol{x}) = 0 \tag{1.6}$$

and the orthonormality conditions

$$\int f_{klm}(\boldsymbol{x})\, f_{k'l'm'}(\boldsymbol{x})\, \mathrm{d}^3x = \delta_{ll'}\,, \delta_{mm'}\,, \delta(k - k')\,. \tag{1.7}$$

In particular

$$f_{k00} = \frac{1}{(2\pi^2)^{\frac{1}{2}}}\, \frac{\sin kr}{r}\,,$$

$$f_{klm} = \left(\frac{3}{2\pi^2}\right)^{\frac{1}{2}} \frac{x_m}{kr^3}\, (\sin kr - kr \cos kr)\,.$$

The meson field can be expanded in terms of the eigenfunctions U_{klm}

$$\Phi_\alpha(\boldsymbol{x}) = \sum (2\omega_k)^{-\frac{1}{2}} [a_{\alpha lm}(k) + a^+_{\alpha lm}(k)]\, f_{klm}(\boldsymbol{x})\,, \tag{1.8}$$

$$\pi_\alpha(\boldsymbol{x}) = \sum \left(\frac{\omega_k}{2}\right)^{\frac{1}{2}} [a_{\alpha lm}(k) - a^+_{\alpha lm}(k)]\, f_{klm}(\boldsymbol{x})\,. \tag{1.8'}$$

$a_{\alpha lm}(k)$ and $a^+_{\alpha lm}(k)$ are destruction and creation operators of mesons in the state (k, α, l, m) (*).

Let us now substitute the expansions (1.8) into Eqs. (1.1) and (1.5). It is easy to verify that, as already stated, only p-waves give a non-vanishing contribution to H'. Our total p-wave Hamiltonian becomes (**)

$$H = \sum \omega_k a^+_{\alpha m}(k)\, a_{\alpha m}(k) + f_0 \sum \tau_\alpha \sigma_m [a_{\alpha m}(k) + a^+_{\alpha m}(k)]\, C(k)\,, \tag{1.9}$$

where

$$\omega_k = \sqrt{k^2 + \mu^2}\,, \qquad C(k) = \frac{U(k) k^2}{(3\pi\omega_k)^{\frac{1}{2}}}\,, \qquad U(k) = \int U(\boldsymbol{x}) \exp[i\boldsymbol{k}\boldsymbol{x}]\, \mathrm{d}^3x\,. \tag{1.10}$$

(*) $a_{1lm}(k) - ia_{2lm}(k)$ operates the destruction of a π^+; $a_1 + ia_2$ the destruction of a π^-; $a_1^+ \mp ia_2$ the creation of a π^- and π^+ respectively.

(**) Since in the following we have only to deal with p-wave mesons we suppress the index 1 in all our notations.

As the emission and absorption of low-energy pions should not be affected by the cut-off function, it is therefore necessary that $U(k) \sim 1$, at least for k smaller than 4μ.

At the present time, no reliable method is known for solving the Hamiltonian (1.9). The perturbation-method used in electrodynamics cannot be applied to our case because the coupling of the meson field with the nucleon is too large.

The calculation of pion-nucleon scattering based on that method gives results in violent disagreement with experiment. Improved approximation like the Tamm-Dancoff method has given results in qualitative agreement with experiment, but it has not been possible to evaluate the error introduced by the approximation. We shall here discuss an approach to the static theory, which does not pretend to deduce the scattering cross-sections in terms of the parameters f_0 and $U(k)$ of the theory, but which uses the Hamiltonian (1.9) in order to correlate some experimental results. We shall see that this approach is indeed successful. It allows one to obtain general relations between matrix amplitudes for pion-nucleon scattering and photoproduction of mesons. These relations have been successfully tested against experiments.

2. – The physical nucleon.

The Hamiltonian (1.5) (or (1.9)) describes the system of the source interacting with the meson field. The eigenstates of H describe physical states, *i.e.* states which can be obtained by performing actual experiments.

If one were able to calculate the matrix elements of all meson field operators between those states, one could, in principle, predict any physical process. It is to be noted that in those physical states the number of mesons is not fixed (this comes from the fact that the operators $N_{m\alpha}(k) = a^*_{m\alpha}(k)a_{m\alpha}(k)$, representing the number of mesons in the state k, m, α, does not commute with the total Hamiltonian). Thus it follows that those physical states *cannot* be identified by giving the number of mesons with different k, m, α.

As we shall discuss in the next section, these states will be identified from the asymptotic behaviour of the state vector, *i.e.* from the form of this state vector at a large distance from the source.

In this Section we want to give our attention to the so-called « bound states », *i.e.* those states whose state vector does not contain asymptotically any mesons.

We shall assume that there is *only one such bound state*: *the physical nucleon* having $J = \frac{1}{2}$, $T = \frac{1}{2}$. We shall denote this state by $\psi_{r\sigma}$, $(r, \sigma = 1, 2)$. (In many calculations these indices will be omitted.)

This state will correspond to an eigenvalue E_0 of the total Hamiltonian which is not actually observable (since, of course, nobody has yet been able to measure the difference between the masses of the physical and of the bare nucleon!!). In order to simplify the notation, in the following we shall use H defined as $H^0+H'-E^0$; so we have

$$H\psi = 0\,. \tag{2.1}$$

First we want to consider how strongly the physical nucleon interacts with the meson field. A measure of this strength can be obtained by considering the expectation value, in the physical nucleon state, of the operator $f_0\sigma_m\tau_\alpha$ which is the essential part of (1.9). From invariance under the groups of rotations in the ordinary space and in the isospin space, it can be shown (see Appendix I) that these expectation values can be written as

$$f_0(\psi_{r\sigma},\, \sigma_m\tau_\alpha\psi_{r'\sigma'}) = f\sigma^m_{rr'}\tau^\alpha_{\sigma\sigma'}\,, \tag{2.2}$$

where the matrix elements $\sigma^2_{rr'}$ appearing in the l.h.s. are the usual Pauli matrix elements between the states r and r'. For example, $\sigma^3_{11}=1$, $\sigma^3_{12}=0$.

The constant f is called the *renormalized coupling constant* (to be distinguished from the unrenormalized coupling constant f_0 giving the interaction of the source or « *bare nucleon* » with the meson field). The unrenormalized coupling constant will be of fundamental importance throughout our treatment. We will show that the theoretical analysis of different experimental data leads to a unique value $f^2 = 0.08$.

The physical nucleon can be described as being composed of the bare nucleon surrounded by a meson cloud, *i.e.* by a region in which the probability of finding one or many mesons is not much smaller than 1.

In order to have a better idea of the meaning of the renormalized coupling constant and its intimate connection with the state of the meson cloud, it is interesting to calculate the expectation value of the meson field in the physical nucleon state.

Let us first consider the identity:

$$(\psi,\, [H,\, \pi_\alpha(x)]\psi) = (\psi,\, H\pi_\alpha\psi) - (\psi,\, \pi_\alpha H\psi) = 0\,. \tag{2.3}$$

The left-hand side of this expression can be simply evaluated by using the explicit form of H and the commutation relations (1.2):

$$[H_0,\, \pi_\alpha(x)] = \frac{1}{2}\sum_{\alpha'}\int[(\nabla'\Phi_{\alpha'}(x'))^2,\, \pi_\alpha(x)]\,\mathrm{d}^3x' + \frac{\mu^2}{2}\sum_{\alpha'}\int[\Phi^2_{\alpha'}(x'),\, \pi_\alpha(x)]\,\mathrm{d}^3x' =$$

$$= \frac{1}{2}\sum_{\alpha'}\int 2[\nabla'_i\Phi_{\alpha'}(x'),\, \pi_\alpha(x)]\nabla_i\Phi_{\alpha'}(x')\,\mathrm{d}^3x' + \frac{\mu^2}{2}\sum_{\alpha'}\int 2\Phi_\alpha(x')i\delta_{\alpha\alpha'}\delta(x-x')\,\mathrm{d}^3x'\,.$$

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A partial integration in the first term gives:

$$[H_0, \pi_\alpha(x)] = -i\Delta_2 \Phi_\alpha(x) + i\mu^2 \Phi_\alpha(x)\,.$$

Similarly we obtain

$$[H', \pi_\alpha(x)] = -i(4\pi)^{\frac{1}{2}} \frac{f_0}{\mu} (\psi, \sigma_i \tau_\alpha \psi) \frac{\partial U}{\partial x_i}\,.$$

So finally we obtain from (2.3)

$$(\nabla^2 - \mu^2)(\psi, \Phi_\alpha(x)\psi) = -(4\pi)^{\frac{1}{2}} \frac{f_0}{\mu} (\psi, \sigma_i \tau_\alpha \psi) \frac{\partial U}{\partial x_i}(|x|)\,.$$

Using Eq. (2.2) we get:

$$(\nabla^2 - \mu^2)(\psi, \Phi_\alpha(x)\psi) = -(4\pi)^{\frac{1}{2}} \frac{f}{\mu} \sigma_i \tau_\alpha \frac{\partial U}{\partial x_i}\,. \tag{2.4}$$

Equation (2.4) can be integrated by using the well-known Yukawa Green-function $(1/4\pi)(\exp[-\mu r]/r)$ giving:

$$(\psi, \Phi_\alpha(x)\psi) = (4\pi)^{-\frac{1}{2}} \frac{f}{\mu} \sigma_i \tau_\alpha \int \frac{\exp[-\mu|x-x'|]}{|x-x'|} \frac{\partial U(x')}{\partial x'_i} = \tag{2.5}$$

$$= (4\pi)^{-\frac{1}{2}} \frac{f}{\mu} \sigma_i \tau_\alpha \frac{\partial}{\partial x_i} \int \frac{\exp[-\mu|x-x'|]}{|x-x'|} U(x')\,\mathrm{d}^3x'\,,$$

or, in terms of creation operators,

$$(\psi_r a^+_{\alpha m}(k) \psi_{r'}) = f \frac{C(k)}{\omega_k} \tau_\lambda \sigma_m\,, \tag{2.5'}$$

where $C(k)$ is defined in Eq. (1.10).

Equation (2.5) is a rigorous consequence of our starting Hamiltonian. It is formally identical with the result of semi-classical calculations (see Professor WEISSKOPF's lectures at CERN). The only difference is that the renormalized coupling constant f appears. Physically this is due to the fact that the field is created by the operators $\sigma\tau$ of the bare nucleon. Taking the expectation value of this operator in the physical nucleon state ψ, gives rise to an averaging which reduces the effect of a constant scalar factor (f/f_0 is smaller than 1).

For values of $|x|$ much larger than the cut-off value R, the r.h.s. of (2.5) is practically independent of the form of the cut-off function and can be well

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approximated by setting $U(x) = \delta(x)$

$$(\psi,\ \Phi_\alpha(x)\psi) \approx -(4\pi)^{-\frac{1}{2}}\frac{f}{\mu}\,\tau_\alpha\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\left(\frac{\exp[-\mu r]}{r}\right). \tag{2.6}$$

The theoretical result (2.6) can indeed be tested against experiment. The value of the meson field created by our nucleon can be measured by putting a test-body in such a field and observing the action of the field on the test-body.

We can use as test-body another nucleon located at point x with state vector $\psi'\rangle$, spin σ', isospin τ'. (The first nucleon is, of course, located at the origin.) The distance x between the two nucleons must be taken sufficiently large in order that

a) Eq. (2.6) is valid;

b) the presence of the new nucleon does not perturb the meson cloud of the first one.

The coupling between this new nucleon and the meson field is given by:

$$H' = (4\pi)^{\frac{1}{2}} f_0 \sum_\beta \tau'_\beta \boldsymbol{\sigma}'\cdot\int \boldsymbol{\nabla}\Phi_\beta(x')\,U(x'-x)\,\mathrm{d}^3x'\,.$$

We are interested in the action of the field produced by the nucleon $\psi\rangle$ on the nucleon $\psi'\rangle$. Therefore we take the expectation value of $\boldsymbol{\sigma}'\tau'_\beta$ in the state $\psi'\rangle$ and the expectation value of $\Phi_\beta(x)$ in the state $\psi\rangle$. Using equation (2.6), we obtain for the nucleon-nucleon interaction:

$$W = (4\pi)^{\frac{1}{2}} f\tau'_\beta\boldsymbol{\sigma}'\cdot\int\boldsymbol{\nabla}_{x'}(\psi,\ \Phi_\beta(x')\psi)\,U(x'-x)\,\mathrm{d}^3x' \sim (4\pi)^{\frac{1}{2}} f\tau'_\beta\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}_{x'}(\psi,\ \Phi_\beta(x)\psi) = \tag{2.7}$$

$$= f^2(\tau_\alpha\tau'_\alpha)(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}'\cdot\boldsymbol{\nabla})\left[\frac{\exp[-\mu|x|]}{|x|}\right].$$

3. – The scattering state.

In the last Section we have discussed the general properties of the lowest eigenstate of H, the physical nucleon; we now want to consider the higher energy eigenstates $\psi_{sc}(\omega_0)$ representing physical pion-nucleon scattering states. According to assumption, this is the state of a pion-wave interacting with the fixed nucleon:

$$H\psi_{sc} = \omega_0\psi_{\varepsilon c}\,.$$

As in the usual scattering theory, those states will be identified by their asymptotic form. There is, however, a fundamental difference with the simple

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potential scattering theory: at large distance from the source, the scattering state does not approach any eigenstate of H_0 but tends to a state containing one or many free mesons *plus the physical nucleon*. In this Section we shall limit our attention to one-meson scattering states, *i.e.* to states which asymptotically contain only one meson.

In order to give a sufficiently simple treatment of the scattering problem we shall start from the assumption that in the scattering process the state of the *physical nucleon is not essentially perturbed by the incoming meson*. In other words, we assume that the binding energy of the physical nucleon is much larger than its interaction with the incoming pion. Our assumption leads to the following choice for the pion-nucleon scattering state vector:

$$\psi_{\mathrm{sc}} = \sum_{kr'} \left(\chi_{r'\alpha'm'}(q')a^{+}_{\alpha'm'}(q') + \alpha_{r'}\right)\psi_{r'} \tag{3.1}$$

($\psi_{r'}$ are the four physical nucleon states.)

The use of the momentum representation is suggested for the sake of simplicity, mainly due to the fact that we have to deal with a single value of π.

We remark that Eq. (3.1) properly describes the physical situation when the pion is sufficiently far from the nucleon.

The results we shall obtain are essentially the same as those which could be obtained by a more sophisticated treatment, such as dispersion theory. The quantities χ and α have to be obtained by taking (3.1) as an approximate eigenstate of the total Hamiltonian. In order to do so, we shall use a variational method which will give us the « best » values of such quantities.

Let us consider the matrix element

$$I = (\psi_{\mathrm{sc}}, (H-\omega_0)\psi_{\mathrm{sc}}) . \tag{3.2}$$

The variation δI corresponding to a variation $\delta\psi$ of ψ

$$\delta I = (\psi_{\mathrm{sc}}, (H-\omega_0)\,\delta\psi_{\mathrm{sc}}) , \tag{3.2'}$$

is zero when ψ is an eigenstate of H with an eigenvalue ω_0. We shall therefore insert the « ansatz » (3.1) into (3.2) and ask that (*)

$$\frac{\delta I}{\delta\chi} = 0 , \qquad \frac{\delta I}{\delta\alpha} = 0 .$$

Such a procedure will finally lead us to the equations

$$(\psi_r \psi_{\mathrm{sc}}) = 0 \tag{3.3}$$

$$(\psi_r, a^{+}_{\alpha m}(q)(H-\omega_0)\psi_{\mathrm{sc}}) = 0 . \tag{3.4}$$

(*) The variations of ψ and $\overline{\psi}$ in (3.2) have to be taken independently.

Introducing Eq. (3.1) into (3.3) and (3.4) we have:

$$\alpha_r + \sum_{r'\alpha'm'q'} (\psi_r,\, a^+_{\alpha'm'}(q')\psi_{r'})\,\chi_{r'\alpha'm'}(q') = 0\,, \tag{3.3'}$$

$$\omega_0\chi_{r\alpha m}(q) - \sum_{r'q'} (\psi_r\,[a^+_{\alpha m}(q)(H-\omega_0)a_{\alpha'm'}(q')]\psi_{r'})\,\chi_{r'\alpha'm'}(q') + \\ + \omega_0 \sum_{r'} \alpha_{r'}(\psi_r,\, a_{\alpha m}(q)\psi_{r'}) = 0\,. \tag{3.4'}$$

The second term in the left-hand side of Eq. (3.4′) can be put in a more convenient form using the identity

$$[a(q)[H,\, a^+(q')]] \doteq \omega_q\delta_{qq'}\,,$$

which leads to

$$\langle\psi_r,\, a_{\alpha m}(q)(H-\omega)a^+_{\alpha'm'}(q')\psi_r\rangle = \\ = \delta_{qq'}(\omega_q-\omega) - \langle\psi_r a^+_{\alpha'm'}(q')(H+\omega)a_{\alpha m}(q)\psi_{r'}\rangle\,. \tag{3.5}$$

The simple transformation (3.5) is the essential point in our whole derivation. The matrix element on the left-hand side of Eq. (3.5) has the advantage of having the meson field operators « well ordered » in such a way that only destruction operators operate on the physical nucleon. In this way, if we insert between a^+ and a the complete set of eigenstates of H, the role of the states, other than the physical nucleon, is comparatively unimportant, because a on the physical nucleon can only destroy a pion in the cloud and this has little effect at a large distance from the nucleon. The same would not be true if we directly inserted the complete set of states in the left-hand side of (3.5). Thus recalling that (see Eq. (2.5′))

$$(\psi_r a^+_{\alpha m}(q)\psi_{r'}) = if\,\frac{C(q)}{\omega_q}\,\tau_\alpha\boldsymbol{\sigma}_m\,, \tag{3.6}$$

$$(\psi_r a^+_{\alpha m}(q)(H-\omega_0)a_{\alpha'm'}(q')\psi_{r'}) = \\ = (\omega_q-\omega_0)\delta_{\alpha\alpha'}\delta_{mm'}\delta_{qq'} - C(q)C(q')f^2\sigma_{m'}\sigma_m\tau_{\alpha'}\tau_\alpha\,. \tag{3.6'}$$

Introducing (3.5) and (3.6) into (3.3′) and (3.4′) we obtain the following integral equation for χ:

$$(\omega_k-\omega_0)\chi_{m\alpha r}(q) - \int K_{\alpha m r,\alpha'm'r'}(q,\, q')\,\chi_{\alpha'm'r'}(q')\,\mathrm{d}q'\,, \tag{3.7}$$

where

$$K_{\alpha m,\alpha'm'r'}(q,\, q') = \omega_0\,\frac{C(q)C(q')}{\omega_q\omega_{q'}}\,f^2\langle r|-\tau_\alpha\tau_{\alpha'}\,\sigma_m\sigma_{m'} + \tau_{\alpha'}\tau_\alpha\sigma_{m'}\sigma_m|r'\rangle\,. \tag{3.8}$$

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Let us consider Eq. (3.7). The left-hand side represents the free propagation of the meson; indeed, if K were zero, the solution of (3.7) would be of the form

$$\chi = \delta(\omega_q - \omega_0)$$

which represents a free meson-wave with energy ω_0.

The effect of the « potential term » in K is to produce a deviation of χ from the simple δ-function, *i.e.* to give rise to scattering.

The potential term (3.8) can be expanded in terms of the projection operators on the different scattering states (see Appendix III)

$$K = f^2 \frac{C(q)\,C(q')}{\omega_q \omega_{q'}}\, \omega_0 (8P_{11} + 2(P_{13} + P_{31}) - 4P_{33})\,. \tag{3.9}$$

The coefficients of the different projection operators give the potential acting of the wave-functions relative to the different values of J and T. By expanding the wave-function χ in a similar way

$$\chi_{\tau\alpha m} = \sum \chi_i P_i \tag{3.9'}$$

we get

$$(\omega_q - \omega_0)\,\chi_i(q) = \lambda_i f^2 \omega_0 \int \frac{C(q)\,C(q')}{\omega_q \omega_{q'}}\, \chi_i(q')\, \mathrm{d}q'\,, \tag{3.10}$$

where the index i refers to the different channels (11; 13; 31; 33)

$$\lambda_i = (-8,\ -2,\ -2,\ -4)\,.$$

From Eq. (3.10), dividing by $(\omega_q - \omega_0)$ we get

$$\chi_i(q) = \delta(\omega_q - \omega_0) + \frac{\lambda_i \omega_0}{\omega_q - \omega_0}\, f^2\, \frac{C(q)}{\omega_q}\, L_i\,, \tag{3.11}$$

$$L_i = \int \frac{C(q')}{\omega_{q'}}\, \chi_i(q')\, \mathrm{d}q'\,. \tag{3.12}$$

The step from Eq. (3.10) to (3.11) is analogous to what one does in solving an inhomogeneous differential equation (the right-hand side of (3.10) is the inhomogeneity): in this case one adds the general solution of the differential equation (the δ-function term) to a solution of the inhomogeneous one. The cohordinate space equivalent of Eq. (3.10) is, indeed, a differential equation, the equivalent of (3.11) and integral equation ($1/(\omega_q - \omega_0)$ is the Fourier transform of the Green-function).

Equation (3.11) gives the function χ_i in terms of the unique number L_i. Such a number can be determined by substituting χ_i into (3.12). One gets

$$L_i = \int \frac{C(q)}{\omega_q}\,\delta(\omega_q - \omega_0)\,\frac{\mathrm{d}q}{\mathrm{d}\omega_q} + \omega_0 \int \frac{C^2(q)\,\mathrm{d}q}{(\omega_q - \omega_0)\omega_q^2}\,L_i\,,$$

and using the expressions for $C(q)$

$$L_i = \frac{q_0/\sqrt{3\pi\omega_0}}{1 - \lambda_i(\omega_0/3\pi)f^2\int_1^\infty q^4u^2(q)\,\mathrm{d}q/(\omega_q - \omega_0)\omega_q^3}\,, \qquad q_0 = \sqrt{\omega_0^2 - \mu^2}\,. \tag{3.12'}$$

Let us now consider the asymptotic form of Eq. (3.11). In order to see it clearly one would have to transform (3.11) in cohordinate space. This transformation is not difficult and can be found in many textbooks on quantum mechanics.

The term $\delta(\omega_q - \omega_0)$ is transformed into $(\sin qr)/r$, the term $1/(\omega_q - \omega_0)$ into $\pi(\cos q_r/r)$. The asymptotic form of Eq. (3.11) will be

$$\frac{\sin qr}{r} + \left(\pi\omega_0\lambda_i f^2\,\frac{C(q_0)}{\omega_0}\,L_i\right)\frac{\cos qr}{r}\,.$$

By assuming

$$\operatorname{tg}\delta_i = \pi\omega_0\lambda_i f^2\,\frac{C(q_0)}{\omega_0}\,L_i\,, \tag{3.13}$$

such an asymptotic form becomes

$$\frac{1}{r\cos\delta}\sin(qr + \delta)\,.$$

This shows that the angle δ defined in (3.13) is the scattering phase-shift. Using Eq. (3.12) we obtain the final result

$$\operatorname{tg}\delta_i(\omega_0) = \frac{(\lambda_i/3)f^2(q_0^3/\omega_0)}{1 - \lambda_i\omega_0 f^2 I(\omega_0)}\,, \tag{3.13'}$$

where

$$I(\omega_0) = \frac{1}{3\pi}\int \frac{q^4\omega^2(q)\,\mathrm{d}q}{(\omega_q - \omega_0)\omega_q^3}\,. \tag{3.14}$$

The integral $I(\omega_0)$ is always positive. It is dominated by the momenta near the cut-off and therefore for $\omega_0 < 2\mu$ it is almost independent of ω_0 (*effective range approximation*).

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Therefore, recalling the values of λ_n we see that the denominator of Eq. (3.13) can vanish only in the case of the 33 phase-shift which will therefore pass through 90°. Eq. (3.13) does indeed explain the existence of the well-known 33 resonance which dominates the whole low-energy pion physics.

Equation (3.13) can be rewritten as follows:

$$\frac{q^3}{\omega}\,\mathrm{ctg}\,\delta_i = \frac{1}{\lambda_i f^2}\,(1-\lambda_i f^2 I\omega)\,. \tag{3.13''}$$

It is interesting to compare this result with that of the Born approximation. The Born phase-shifts will be

$$\delta_{\alpha\,\mathrm{Born}} = \frac{q^3}{\omega}\,\lambda_\alpha\,,$$

and the ctg formula corresponding to (3.14) will be

$$\frac{q^3\,\mathrm{ctg}\,\delta_{\alpha\,\mathrm{Born}}}{\omega} \simeq \frac{1}{\lambda_\alpha}\,.$$

A comparison with (3.14) shows that the main difference is for $\alpha = 3$ as soon as $\omega \gg 1$. One may realize that this is connected to the full symmetry of the H with respect to the two nucleon variables, $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$. Thus the existence of the $\frac{3}{2}\,\frac{3}{2}$ resonant state is in the nature of a check of the model. But instead, the position of the resonance is completely determined by the choice of the cut-off, as one realizes by inspection of (3.14) which defines the integral $I(\omega_0)$.

One has checked that the low-energy experimental data for $(q^3/\omega)\,\mathrm{ctg}\,\delta_{33}$ lie nicely on a straight line. The interception of such a line with the $\omega = 0$ axis gives $f^2 \approx 0.08$ in agreement with the other determinations.

Before closing this Section we wish to derive an identity for the « wave-function » χ which will be useful in the study of photo-meson production. Using Eqs. (3.12), (3.13) and (3.9′) we obtain

$$\int \frac{C(q')}{\omega_{q'}}\,\chi(q')\,\mathrm{d}q' = \frac{1}{\pi C(q_0) f^2}\sum_i \frac{\mathrm{tg}\,\delta_i(\omega_0)}{\lambda_i}\,P_i\,. \tag{3.15}$$

4. – Something about the electromagnetic properties of the nucleon.

Before studying photomeson production we have to briefly discuss the static electromagnetic properties of the nucleon.

The general form of the interaction between the nucleon and the electromagnetic field $\boldsymbol{A}(x)$, $\Phi(x)$ is

$$H_{\text{im}}(x) = -\int \boldsymbol{j}(x)\,\boldsymbol{A}(x) + \int \varrho(x)\,\Phi(x)\,\mathrm{d}^3x\,. \tag{4.1}$$

The operators $\varrho(x)$ and $\boldsymbol{j}(x)$ represent the total charge and current density of the system composed by the nucleon source and the meson field.

The law of conservation of electric charge requires that

$$\operatorname{div}\boldsymbol{j}(x) = \frac{\mathrm{d}\varrho}{\mathrm{d}t} = i[\varrho(x), H]\,. \tag{4.2}$$

We want to consider here an electromagnetic field corresponding to a photon of momentum $\boldsymbol{k}$

$$\begin{cases} \boldsymbol{A}(x) = \boldsymbol{A}\exp[i\boldsymbol{k}\cdot\boldsymbol{x}] \\ \Phi(x) = \Phi\exp[i\boldsymbol{k}\cdot\boldsymbol{x}] \end{cases} \tag{4.3}$$

which causes a transition between physical nucleon states. (Such a situation is encountered, for example, in the case of electron-nucleon scattering.) The matrix elements for such a transition are

$$\int(\psi_i, \varrho(x)\psi_i)\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,\mathrm{d}^3x = \varrho_i(k), \tag{4.4}$$

$$\int(\psi_i, \boldsymbol{j}(x)\psi_i)\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,\mathrm{d}^3x = \boldsymbol{j}_i(k)\,. \tag{4.5}$$

The index i distinguishes between proton and neutron, while $\varrho_i(k)$, $\boldsymbol{j}_i(k)$ are still operators in spin variables.

For $k \to 0$ the left-hand side of Eq. (4.4) reduces to the total charge, so

$$\begin{cases} \varrho_{\text{P}}(0) = e \\ \varrho_{\text{N}}(0) = 0\,. \end{cases} \tag{4.6}$$

Because ϱ_i must be invariant with respect to rotations and inversions, the most general form of ϱ_i is given by

$$\begin{cases} \varrho(k^2) = e\,F^{(1)}_{\text{P}}(k^2) \\ \varrho(k^2) = e\,F^{(1)}_{\text{N}}(k^2) \end{cases} \tag{4.7}$$

where

$$F^{(1)}_{\text{P}}(0) = 1\,, \qquad F^{(1)}_{\text{N}}(0) = 0\,.$$

(Invariance under rotation will permit a term $F^{(1)}(k^2)\boldsymbol{\sigma}\cdot\boldsymbol{k}$; this is, however, to be discarded because $\boldsymbol{\sigma}\cdot\boldsymbol{k}$ is a pseudoscalar.)

Let us now consider Eq. (4.5). For invariance reasons, the most general form for $\boldsymbol{j}_i(\boldsymbol{k})$ is

$$\boldsymbol{j}_i(\boldsymbol{k}) = \alpha(k^2)\boldsymbol{k} + i\mu(k^2)(\boldsymbol{k}\times\boldsymbol{\sigma}) \tag{4.8}$$

(a term in $\boldsymbol{\sigma}$ is not allowed because $\boldsymbol{\sigma}$ is a pseudovector. Instead $\boldsymbol{\sigma}\times\boldsymbol{k}$ is a vector).

Equation (4.8) is written with an i in the second term in order that $\mu(k^2)$ be real. The reality of α and μ is proved by applying time reversal (*).

We now want to prove that $\alpha(k^2)=0$ and $\boldsymbol{j}_i(\boldsymbol{k})$ simply reduces to $i\mu(k^2)(\boldsymbol{k}\times\boldsymbol{\sigma})$.

Using Eq. (4.2) one gets

$$\begin{aligned} (k\cdot j_i(k)) &= -i\int\sum_\alpha(\psi_i, \boldsymbol{j}_\alpha(x)\psi_i)\,\frac{\partial\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]}{\partial x_\alpha}\,\mathrm{d}^3x = \\ &= i\int(\psi_i, \operatorname{div}\boldsymbol{j}\psi_i)\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,\mathrm{d}^3x = \int(\psi_i, [H, \varrho(x)]\psi_i)\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,\mathrm{d}^3x = 0\,, \end{aligned} \tag{4.9}$$

because $H\psi_i = 0$.

Then Eq. (4.8) gives

$$\boldsymbol{k}\cdot\boldsymbol{j}_i(k) = \alpha(k^2)k^2 = 0\,. \qquad q.e.d.$$

The function $\mu(k^2)$ can be interpreted by considering the interaction

$$\boldsymbol{j}_i(k)\cdot A = i\mu_i(k^2)(\boldsymbol{k}\times\boldsymbol{\sigma})\cdot\boldsymbol{A} = i\mu_i(k^2)\boldsymbol{\sigma}\cdot\boldsymbol{k}\times\boldsymbol{A} = \mu_i(k^2)(\boldsymbol{\sigma}\cdot\boldsymbol{H})\,, \tag{4.9'}$$

where $\boldsymbol{H}$ is the external magnetic field.

So $\mu_i(k^2)$ is the magnetic moment of the physical nucleon « seen » by a photon of momentum $\boldsymbol{k}$.

In the limit $k\to 0$ they will coincide with the *static magnetic moments of the nucleons*

$$\mu_i(k^2) = \mu_i F_i^{(2)}(k^2) \tag{4.10}$$

with $F_i^{(2)}(0)=1$.

(*) Actually, under time reversal $\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{\sigma}$ must change sign and every complex number is transformed into its conjugate. Thus we have

$$-[\alpha^*(k^2)(-\boldsymbol{k}) + \mu^*(k^2)(-i)(-\boldsymbol{k}\times-\boldsymbol{\sigma})] = \alpha(k^2)\boldsymbol{k} + i\mu(k^2)(\boldsymbol{k}\times\boldsymbol{\sigma})\,,$$

so

$$\alpha^*(k^2) = \alpha(k^2) \qquad \text{and} \qquad \mu^*(k^2) = \mu(k^2)\,.$$

The four functions $F(k^2)$ appearing in Eqs. (4.7) and (4.10) are the static limits of the electromagnetic form factors measured by Hofstadter in electron-proton and electron-deuteron experiments.

Introducing the giromagnetic ratios g_p, g_N of the nucleons, Eq. (4.8) can be rewritten (neglecting the k^2 dependence of the form factors)

$$j(k) = i\left[\left(\frac{g_p + g_N}{4M}\right) + \left(\frac{g_p - g_N}{4M}\right)\tau_3\right]\sqrt{4\pi}\, e(k\times\sigma)\,. \tag{4.10'}$$

5. – Photoproduction of pions.

The considerations developed in Section **4** are essential when making a rather simple process: the photoproduction of pions, which at first looks rather complicated.

In a more or less intuitive way we may consider that photoproduction takes place mainly in three different ways.

a) The photon is absorbed by the nucleon which subsequently emits the pion in a resonant p-33 state: it is not surprising that this «nucleon term» is going to depend on the static magnetic moment of the nucleon.

b) The photon is absorbed by a pion of the nucleon cloud, which in the collision receives the energy and momentum necessary to become a free pion (photo-electric effect). This process takes place at a large distance from the core and will contain all possible one-pion waves. It will not be, by any means, dominated by the 33 resonance.

c) In the emission of a π^+ pion, the nucleon changes from proton to neutron. This sharp variation of the nucleon charge due to this emission can lead to the absorption of the photon. This is a typical core effect and therefore takes place in the lowest possible multipole: the electric dipole. The meson will be emitted in a s-wave. The sign of this term, which in the relativistic theory is associated with the creation of a nucleon-antinucleon pair, has been first computed rigorously by KROLL and RUDERMAN using gauge invariance arguments.

In a quite general manner, the interaction operator can be written: $\boldsymbol{j}\cdot\boldsymbol{A}$. Thus the photoproduction matrix element is given by the matrix element of the current between a pion-nucleon scattering state and the physical nucleon

$$H = \frac{1}{(2k)^{\frac{1}{2}}}\int(\psi_{\text{scatt}}, \boldsymbol{j}(x)\boldsymbol{\epsilon}\psi_0)\exp[i\boldsymbol{kx}]\,\mathrm{d}^3x\,; \tag{5.1}$$

k and $\boldsymbol{\epsilon}$ are the photon momentum and polarization.

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In order to evaluate the photoproduction matrix element (5.1) we have to use an explicit form for the scattering state vector.

We shall assume that pions are scattered in p-waves. All other waves do not suffer any appreciable scattering.

Let us first consider photoproduction of pions in p-waves. Those matrix elements are obtained in eq. (4.1) in the form of the scattering state vector given in eq. (3.1):

$$\psi^{p}_{sc} = \sum_{\alpha' m'} (\chi_{\alpha' m'}(q') a^{+}_{\alpha' m'}(q') + \alpha)\psi_0 ,$$

where all symbols are defined in Section **3**.

We obtain

$$(5.2)\qquad H = \frac{1}{(2k)^{\frac{1}{2}}}\int d^3x \exp[ikx]\boldsymbol{\epsilon}\cdot\sum_{\alpha' m' q'}\{(\psi_0 a_{\alpha' m'}(q')\boldsymbol{j}(x)\psi_0) - - (\psi_0 \boldsymbol{j}(x)\psi_0)(\psi_0 a_{\alpha' m'}(q')\psi_0)\}\chi_{\alpha' m'}(q') .$$

We are led to the evaluation of some matrix elements in the physical nucleon state. The second term in the parenthesis can be simply obtained in terms of the nucleon magnetic moment using Eq. (4.10′). For the first term we commute a and j and then insert between the two operators a complete set of eigenstates of the total Hamiltonian.

As in the scattering case, the dominant contribution will come from the physical nucleon states; we can therefore write

$$(5.3)\qquad (\psi a_{0\alpha' m'}(q')\boldsymbol{j}(x)\psi_0) \simeq (\psi_0 \boldsymbol{j}(x)\psi_0)(\psi_0 a_{\alpha' m'}(q')\psi_0) + (\psi_0[a_{\alpha' m'}, j(x)]\psi_0) .$$

The photoproduction matrix element becomes

$$(5.4)\qquad H = \frac{1}{(2k)^{\frac{1}{2}}}\int d^3x \exp[ikx]\boldsymbol{\epsilon}\sum_{\alpha' m' q'}\{(\psi_0 \boldsymbol{j}(x)\psi_0)(\psi_0 a_{\alpha' m'}(q')\psi_0) - - (\psi_0 a_{\alpha' m'}(q')\psi_0)(\psi_0 \boldsymbol{j}(x)\psi_0) + (\psi_0[a_{\alpha' m'}(q'), j(x)]\psi_0)\}\chi_{\alpha' m'}(q') .$$

Equation (5.4) shows clearly the physical features of photoproduction. The first two terms in the parenthesis are related to the absorption of the photon by the physical nucleon, the commutator is connected with the absorption by the meson cloud. The integral on the « wave-function » χ accounts for the rescattering effects, *i.e.* for the effects of the final state pion-nucleon interaction.

Equation (5.4) also shows that the only knowledge of the current operator $\boldsymbol{j}$ needed for an evaluation of photoproduction is about the matrix elements

$$(\psi \boldsymbol{j} \psi) \qquad \text{and} \qquad (\psi, [a, \boldsymbol{j}]\psi) .$$

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The first term is given in Eq. (4.10′) in terms of the observed magnetic moments of the nucleon. The second requires a detailed knowledge of the part of the current operator containing the meson field.

As already discussed in the nucleon term, the meson is emitted in a purely p-wave state, whereas the commutator term represents the p-wave part of the whole effect of the meson cloud for which the meson can be emitted in any angular momentum.

Let us first evaluate the nucleon term which represents the leading contribution to π^0 production.

Recalling Eqs. (3.6) and (4.9) and keeping only the τ_3 part of the magnetic moment, such a term becomes

$$H = \frac{i\sqrt{4\pi}\,ef}{(2k)^{\frac{1}{2}}}\,\frac{g_{\rm P}-g_{\rm N}}{4M}\sum_{\alpha'm'}[\boldsymbol{\sigma\epsilon}\times\boldsymbol{k},\,\tau_{\alpha'}\sigma_{m'}]\int^{\infty}\frac{C(q)}{\omega_q}\,\chi_{\alpha'm'}(q)\,\mathrm{d}q\,, \tag{5.5}$$

and using Eq. (3.15)

$$H = \frac{i\sqrt{4\pi}\,ef}{(2k)^{\frac{1}{2}}}\left(\frac{e}{f}\right)\frac{g_{\rm P}-g_{\rm N}}{4\,M}\sum_{\alpha'm'}\,[\boldsymbol{\sigma\epsilon}\times\boldsymbol{k},\,\tau_{\alpha'}\sigma_{m'}]\sum_{i}\,(m'\alpha'\,|\,\frac{P_i\,\mathrm{tg}\,\delta_i}{\lambda_i}\,. \tag{5.6}$$

We keep nothing but the contribution from the 33 state which is the very dominant term. The magnetic dipole R_{33} reaction matrix element to such a state is simply obtained from (4.6):

$$R_{33} = \frac{g_{\rm P}-g_{\rm N}}{4M}\,\frac{e}{f}\,\frac{\sqrt{6k}}{q^2}\,\mathrm{tg}\,\delta_{33}\big\{2\delta_{\mu3}+i[\tau_\mu\tau_3]\big\}\,. \tag{5.7}$$

The matrix element given in (5.7) is the *reaction* matrix element for photoproduction. We arrived at the reaction matrix and not at the S matrix because our scattering state was defined with stationary boundary conditions. The corresponding S matrix element M_{33} is simply obtained by multiplying (5.7) by $\exp[i\delta]\cos\delta$ (*)

$$M_{33} = \frac{g_{\rm P}-g_{\rm N}}{4M}\,\frac{e}{f}\,\frac{\sqrt{6k}}{q^2}\,\sin\delta_{33}\exp[i\delta_{33}]\,(2\delta_{\mu3}+i[\tau_\mu\tau_3])\,. \tag{5.8}$$

(*) If the asymptotic form in co-ordinate representation of the scattering wave function were

$$\psi \sim \frac{\exp[ikr]-\exp[-ikr]\exp[2i\delta]}{2ir} = \frac{\sin kr+\exp[i\delta]\sin\delta\exp[-ikr]}{r}\,,$$

we would have arrived at the S matrix. Since the asymptotic form used in Sect. **3** is

$$\psi \sim \sin kr + \mathrm{tg}\,\delta\cos kr = \frac{1}{\cos\delta\exp[i\delta]}\left(\frac{\sin kr+\exp[i\delta]\sin\delta\exp[-ikr]}{r}\right),$$

we arrive at the reaction matrix.

Eq. (5.8) gives a simple expression for the nucleon terms which dominate completely the π^0 photoproduction.

Let us now discuss briefly the contribution of the meson currents. As already pointed out, such terms are connected to the commutator of the meson field with $\boldsymbol{j}(x)$. However, if we want to account for the whole effect, we cannot consider the commutator with an operator $a_{\alpha m}(q)$ of a p-wave since the pion can be emitted with any angular momentum. The result for those terms is very simple: it consists simply in the Born approximation given by

$$H^\pi = \frac{\boldsymbol{\epsilon}}{(2k)^{\frac{1}{2}}} \int (\psi_0, [\boldsymbol{j}(x), a_{\boldsymbol{q}}]\psi_0) \exp[i\boldsymbol{k}\boldsymbol{x}] \, \mathrm{d}^3x \,, \tag{5.9}$$

where a_q is the creation operator of a meson of momentum $\boldsymbol{q}$.

Evaluation of this term can be performed, and leads to the well-known result:

$$H^\pi = \frac{ief2\pi}{\sqrt{k\omega}} \frac{[\tau_q, \tau_3]}{2} \left[\boldsymbol{\sigma\epsilon} + 2\,\frac{\boldsymbol{\sigma}(\boldsymbol{k}-\boldsymbol{q})\,\boldsymbol{q\epsilon}}{(k-q)^2+\mu^2}\right]. \tag{5.10}$$

The physical significance of Eq. (5.10) is quite clear. The first term is the effect of the interaction current discussed in *c*). It is proportional to the renormalized coupling constant and its experimental observation leads to a good determination of f. The second term is the proper meson current effect discussed in *b*). There are, of course, corrections to the simple result (5.9) and (4.10) due to final state pion-nucleon interaction. However, those corrections affect only the p-wave part of (5.10) which is not very important. Such corrections can therefore be safely neglected.

The whole π^0 production is therefore given in terms of the magnetic dipole transition given in Eq. (5.8). The total $(\gamma\pi^0)$ cross-section can be obtained by squaring the matrix element (5.8) and summing on the final states. We get

$$\sigma_{\gamma\pi^0} = \frac{32\pi}{9}\frac{k}{q}\left(\frac{g_p - g_N}{4M}\right)^2 \left(\frac{e}{f}\right)^2 \frac{\sin^2\delta_{33}}{q^2} = \frac{k}{q}\left(\frac{g_\nu - g_N}{4M}\right)^2 \left(\frac{e}{f}\right)^2 \sigma_{\pi^0\pi^0}\,. \tag{5.11}$$

The simple expression for the total cross-section has been checked experimentally and has led to another determination of f.

* * *

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APPENDIX I

I.1. – First we want to quickly recapitulate the formulation of charge independence for the pion-nucleon system in terms of rotations in the isospin space. The meson field is strongly coupled to the nucleons, so that the effects of the exchange of one or many mesons can be important in nuclear forces, and we have to impose on our meson-nucleon interaction the condition that it shall not violate the isotopic spin conservation law for nuclear forces. This can be done by assigning an isotopic spin to the meson and requiring the conservation of the sum of the isotopic spin of the nucleons and of the mesons.

The experimental fact that the masses of the three-pion are approximately equal suggests that the one-pion state is a charge triplet. Therefore, we shall assign an isotopic spin one to the pion. In order to write down an explicit expression for such an operator, we have to require that the operators Φ_1, Φ_2, Φ_3 which describe the meson field behave like vector components under rotations in the isotopic spin space, $U_\pi = \exp[i\boldsymbol{\eta}\cdot\boldsymbol{t}]$; that is

$$\exp[+i\boldsymbol{\eta}\cdot\boldsymbol{t}]\boldsymbol{\Phi}\exp[-i\boldsymbol{\eta}\cdot\boldsymbol{t}] = \boldsymbol{\Phi} + \boldsymbol{\eta}\times\boldsymbol{\Phi}\,,$$

where $\boldsymbol{t}$ is the isotopic spin of the meson field, and $\boldsymbol{\eta}$ is a vector with infinitesimal modulus $\mathrm{d}\Phi$ and the direction $\boldsymbol{\eta}$ of the rotation of the axis.

From this formula it follows:

$$[t_\lambda, \Phi_\mu] = i\varepsilon_{\lambda\mu\nu}\Phi_\nu\,.$$

Equivalent relations must also hold the operators $\boldsymbol{\pi}$

$$[t_\lambda, \pi_\mu] = i\varepsilon_{\lambda\mu\nu}\pi_\nu\,.$$

From these relations we can derive:

$$t_i = \varepsilon_{ijk}\int(\Phi_j\pi_k - \Phi_k\pi_j)\,\mathrm{d}^3x\,.$$

Charge independence requires that the three components of the total isotopic spin

$$T_i = \tfrac{1}{2}\tau_i + t_i\,,$$

are constants of the motion. This is equivalent to saying that the theory must be invariant under the group of unitary transformation

$$U = U_{\pi} U_{\mathrm{N}} = \exp\left[i\boldsymbol{T}\cdot\boldsymbol{\eta}\right].$$

Many direct consequences of the charge independence hypothesis for reactions involving pions will be discussed in these lectures.

I.2. – Let us now consider the formula (2.2)

$$(\psi_{rs},\, \sigma_m \tau_\alpha \psi_{r's'}) = C^m_{rr'}\, \sigma\tau^\alpha_{ss'}\,, \tag{AI.1}$$

where $C = f/f_0$

In order to justify Eq. (AI.1) we shall prove the analogous formula of it when only isospins are present:

$$(\psi_s,\, \tau_\alpha \psi_{s'}) = C\tau^\alpha_{ss'}\,. \tag{AI.2}$$

Generalization to (AI.1) will offer no new difficulty.

We recall that the physical nucleon states ψ_{p} and ψ_{N} as eigenstates of the total Hamiltonian are eigenstates of the total isospin defined by

$$\boldsymbol{T} = \tfrac{1}{2}\boldsymbol{\tau} + \boldsymbol{t}\,. \tag{AI.3}$$

This means that

$$T_3\psi_{\mathrm{p}} = \tfrac{1}{2}\psi_{\mathrm{p}}\,, \qquad T_3\psi_{\mathrm{N}} = -\tfrac{1}{2}\psi_{\mathrm{N}}\,, \tag{AI.4}$$

$$\left\{\begin{array}{ll} T^+\psi_{\mathrm{p}} = 0\,, & T^+\psi_{\mathrm{N}} = \psi_{\mathrm{p}}\,, \\ T^-\psi_{\mathrm{p}} = \psi_{\mathrm{N}}\,, & T^-\psi_{\mathrm{N}} = 0\,, \\ T^+ = \dfrac{T_1 + iT_2}{\sqrt{2}} & T^- = \dfrac{T_1 - iT_2}{\sqrt{2}}\,. \end{array}\right. \tag{AI.5}$$

Eq. (AI.4) is a consequence of charge conservation, Eq. (AI.5) of charge independence.

Let us first consider Eq. (AI.4) together with the commutators

$$\begin{array}{c} [T_3, \tau_3] = 0\,, \\ [T_3, \tau^+] = \tau^+\,, \qquad [T_3, \tau^-] = -\tau^-\,. \end{array} \tag{AI.6}$$

Taking the matrix elements of Eq. (AI.6) between ψ_s and $\psi_{s'}$ and using Eq. (AI.4) we get

$$\left\{\begin{array}{l} (s - s')\,(\psi_s,\, \tau_3\,\psi_{s'}) \quad\;\; = 0\,, \\ (s - s' - 1)\,(\psi_s,\, \tau^+\psi_{s'}) = 0\,, \\ (s - s' + 1)\,(\psi_s,\, \tau^-\psi_{s'}) = 0\,, \end{array}\right. \tag{AI.7}$$

(s denotes the eigenvalue of T_3 corresponding to the state ψ_s).

Eq. (AI.7) tells us that the only matrix elements different from zero are

$$(\psi_p, \tau_3\psi_p)\,, \qquad (\psi_N, \tau_3\psi_N)\,,$$
$$(\psi_p, \tau^+\psi_N)\,, \qquad (\psi_N, \tau^-\psi_p)\,.$$

Let us finally use the commutation relation

$$\text{(AI.8)} \qquad \left\{ \begin{aligned} [T^+, \tau_3] &= -\tau^+\,, \\ [T^-, \tau_3] &= \quad \tau^-\,, \\ (\psi_p, [T^+, \tau_3]\psi_N) &= -(\psi_p, \tau^+\psi_N)\,, \\ (\psi_N, [T^-, \tau_3]\psi_p) &= \quad (\psi_N, \tau^-\psi_p)\,. \end{aligned} \right.$$

Using Eq. (AI.5) we obtain

$$\text{(AI.9)} \qquad \left\{ \begin{aligned} (\psi_p, \tau_3\psi_p) &= \quad (\psi_p, \tau^+\psi_N)\,, \\ (\psi_N, \tau_3\psi_N) &= -(\psi_N, \tau^-\psi_p)\,. \end{aligned} \right.$$

Eq. (AI.9) shows that $(\psi_p, \tau^+\psi_N)$ is a real number. Since τ^+ is the hermitian conjugate of τ^- we have

$$\text{(AI.10)} \qquad (\psi_p, \tau^+\psi_N) = (\psi_N, \tau^-\psi_p)\,.$$

In order to summarize all our results, let us define

$$(\psi_p, \tau_3\psi_p) = C\,.$$

Then the matrix elements of the τ matrices are given by

$$\text{(AI.11)} \qquad \left\{ \begin{aligned} (\psi_s, \tau_3\psi_{s'}) &= C\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \\ (\psi_s, \tau^+\psi_{s'}) &= C\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \\ (\psi_s, \tau^-\psi_{s'}) &= C\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \end{aligned} \right.$$

which is the result we wanted to prove.

APPENDIX II

We want to discuss a simple method which allows us to classify the pion-nucleon states according to their angular momenta, spins and isospins.

Let us first consider the spin and angular momentum variables. A complete set of operations for those variables of the nucleon is the unit operator and the three σ_i matrices. Analogously all operators connecting different m states of the pion can be expressed in terms of the unit operator and of the angular momentum operators l_μ of the pion.

Those operators satisfy the canonical commutation rules

$$[l_\mu, l_\nu] = i\varepsilon_{\mu\nu\varrho} l_\varrho \, . \tag{AII.1}$$

The explicit form of the matrix elements of l between wave pion states with different m is (*)

$$\langle m | l_i | m' \rangle = i\varepsilon_{mim'} \, . \tag{AII.2}$$

Of course, the matrix elements of the unit operator are

$$\langle m | 1 | m' \rangle = \delta_{mm'} \, . \tag{AII.2$'$}$$

The only two operators invariant under rotations we can build are therefore 1 and $(\boldsymbol{\sigma}\cdot\boldsymbol{l})$.

Any operator A invariant under the group of rotations can therefore be expanded in the form

$$A = A^+ + A^- \boldsymbol{\sigma}\cdot\boldsymbol{l} \, , \tag{AII.3}$$

(A^+ and A^- are two numbers).

Example: show that

$$\langle m | (1 - \boldsymbol{\sigma}\cdot\boldsymbol{l}) | m' \rangle = \sigma_m \sigma_{m'} \, .$$

Indeed

$$\langle m | (1 - \boldsymbol{\sigma}\cdot\boldsymbol{l}) | m' \rangle = \delta_{mm'} - 2\varepsilon_{mm'i}\sigma_i = \sigma_m \sigma_{m'} \, .$$

An operator invariant under rotations will have as eigenstates states corresponding to a definite value of the momentum J, *i.e.* $J = \frac{1}{2}$, $J = \frac{3}{2}$.

The eigenvalues $A_{\frac{1}{2}}$ and $A_{\frac{3}{2}}$ corresponding to such eigenstates will just be linear combinations of A^+ and A^-. In order to obtain such combinations we shall introduce the projection P_1, P_3 which has the following properties (**)

$$\begin{cases} P_1 | \frac{1}{2} \rangle = | \frac{1}{2} \rangle \, , & P_1 | \frac{3}{2} \rangle = 0 \, , \\ P_3 | \frac{1}{2} \rangle = 0 \, , & P_3 | \frac{3}{2} \rangle = | \frac{3}{2} \rangle \, , \end{cases} \tag{AII.4}$$

where $|\frac{1}{2}\rangle$, $|\frac{3}{2}\rangle$ represent any state with total $J = \frac{1}{2}$ and $\frac{3}{2}$ respectively.

These operators can be constructed explicitly by recalling that

$$J^2 = (\boldsymbol{l} + \tfrac{1}{2}\boldsymbol{\sigma})^2 = \boldsymbol{l}^2 + \frac{\boldsymbol{\sigma}^2}{4} + (\boldsymbol{l}\cdot\boldsymbol{\sigma}) = \frac{11}{4} + (\boldsymbol{l}\cdot\boldsymbol{\sigma}) \, ,$$

has eigenvalues $\frac{1}{2}(\frac{1}{2}+1) = \frac{3}{4}$ and $\frac{3}{2}(\frac{3}{2}+1) = \frac{15}{4}$ in the states $|\frac{1}{2}\rangle$ and $|\frac{3}{2}\rangle$ respec-

---

(*) Eq. (AII.2) can be simply obtained by recalling (see Appendix I) that the operator exp $[i\boldsymbol{l}\cdot\boldsymbol{\eta}\,\mathrm{d}\Phi)$ induces an infinitesimal rotation of an angle $\mathrm{d}\Phi$ around the axis $\boldsymbol{\eta}$.

(**) For simplicity we shall denote by P_1, P_3 the operators $P_{\frac{1}{2}}$, $P_{\frac{3}{2}}$.

tively. So our projection operators will be given by

$$\text{(AII.5)}\qquad \begin{cases} P_1 = \dfrac{\frac{15}{4} - J^2}{\frac{15}{4} - \frac{3}{4}} = \dfrac{1 - (\boldsymbol{l}\cdot\boldsymbol{\sigma})}{3}\,, \\[2ex] P_3 = \dfrac{J^2 - \frac{3}{4}}{\frac{15}{4} - \frac{3}{4}} = \dfrac{2 + (\boldsymbol{l}\cdot\boldsymbol{\sigma})}{3}\,. \end{cases}$$

Conversely we have

$$\text{(AII.5}'\text{)}\qquad P_1 + P_3 = 1\,, \qquad -2P_1 + P_3 = (\boldsymbol{l}\cdot\boldsymbol{\sigma})\,.$$

Introducing (AII.5) into Eq. (AII.3) we have

$$A = (A^+ - 2A^-)P_1 + (A^+ + A^-)P_3\,.$$

Thus we obtain

$$\text{(AII.6)}\qquad A_1 = A^+ - 2A^-\,, \qquad A_3 = A^+ + A^-\,,$$

which is the relation we were looking for.

The formalism is simply generalized to the case where isospins are also present. Let us introduce the isospin t of the meson. All operators invariant under rotation in the spin and isospin space can be written as

$$\text{(AII.7)}\qquad A = A^{++} + A^{+-}\boldsymbol{\sigma}\cdot\boldsymbol{l} + A^{-+}\boldsymbol{\tau}\cdot\boldsymbol{t} + A^{--}(\boldsymbol{\sigma}\cdot\boldsymbol{l})(\boldsymbol{\tau}\cdot\boldsymbol{t})\,.$$

The eigenvalues of A can be obtained by using the projection operators

$$\text{(AII.8)}\qquad \begin{cases} P_{11} = P_1^J P_1^T = \dfrac{(1 - \boldsymbol{\sigma}\cdot\boldsymbol{l})(1 - \boldsymbol{\tau}\cdot\boldsymbol{t})}{9}\,, \\[2ex] P_{13} = P_1^J P_3^T = \dfrac{(1 - \boldsymbol{\sigma}\cdot\boldsymbol{l})(2 + \boldsymbol{\tau}\cdot\boldsymbol{t})}{9}\,, \\[2ex] P_{31} = P_3^J P_1^T = \dfrac{(2 + \boldsymbol{\sigma}\cdot\boldsymbol{l})(1 - \boldsymbol{\tau}\cdot\boldsymbol{t})}{9}\,, \\[2ex] P_{33} = P_3^J P_3^T = \dfrac{(2 + \boldsymbol{\sigma}\cdot\boldsymbol{l})(2 + \boldsymbol{\tau}\cdot\boldsymbol{t})}{9}\,, \end{cases}$$

and we get

$$\text{(AII.9)}\qquad \begin{cases} A_{11} = A^{++} - 2A^{+-} - 2A^{-+} + 4A^{--}\,, \\ A_{13} = A^{++} + A^{+-} - 2A^{-+} - 2A^{--}\,, \\ A_{31} = A^{++} - 2A^{+-} + A^{-+} - 2A^{--}\,, \\ A_{33} = A^{++} + A^{+-} + A^{-+} + A^{--}\,. \end{cases}$$

As an application, let us finally prove that the eigenstates of

$$\text{(AII.10)}\qquad (-\tau_\alpha\tau_{\alpha'}\sigma_m\sigma_{m'} + \tau_{\alpha'}\tau_\alpha\sigma_m\sigma_{m'})\,,$$

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are

$$\lambda_i = (-8,\ -2,\ -2,\ +4)\,.$$

Indeed, using Eq. (AII.3) the operator (AII.10) can be put into the form:

$$-(1-\sigma l)(1-\tau t) + (1+\sigma l)(1+\tau t) = 2(\sigma l + \tau t)\,.$$

Eq. (AII.9) leads to the eigenvalues λ_i.

Example: let us consider the case that all the scattering takes place in a state of isospin $\frac{3}{2}$.

Then the isotopic spin part of the T matrix has the form

$$\langle \alpha r | \frac{2+\boldsymbol{\tau}\cdot\boldsymbol{t}}{3} | \alpha' r' \rangle\,.$$

αr, $\alpha' r'$ denote the initial and final meson and nucleon charge state.

The total cross-section is obtained by squaring the T matrix element, summing over all final states and multiplying by the statistical factor. The total cross-section is therefore proportional to

$$\sum_{\alpha' r'} \langle \alpha r | \frac{2+\boldsymbol{\tau}\cdot\boldsymbol{t}}{3} | \alpha' r' \rangle \langle \alpha' r' | \frac{2+\boldsymbol{\tau}\cdot\boldsymbol{t}}{3} | \alpha r \rangle = \langle \alpha r | \frac{(2+\boldsymbol{\tau}\cdot\boldsymbol{t})(2+\boldsymbol{\tau}\cdot\boldsymbol{t})}{9} | \alpha r \rangle =$$

$$= \langle \alpha r | \frac{2+\boldsymbol{\tau}\cdot\boldsymbol{t}}{3} | \alpha r \rangle = \langle \alpha r | \frac{2+\tau_3 t_3}{3} | \alpha r \rangle\,,$$

since $\tau_1 t_1$ and $\tau_2 t_2$ give 0 if taken between the same states.

Let us now specify that r represents a proton and α either a π^+ or a π^-. Then we have

$$\langle P\pi^+ \left| \frac{2+\tau_3 t_3}{3} \right| P\pi^+ \rangle = \frac{2+1}{3} = 1\,,$$

$$\langle P\pi^- \left| \frac{2+\tau_3 t_3}{3} \right| P\pi^- \rangle = \frac{2-1}{3} = \frac{1}{3}\,.$$

So we obtain that the hypothesis of the dominance of the $T=\frac{3}{2}$ state gives a ratio 3 to 1 of the π^+ and π^- total cross-sections.

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Strange Particles.

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1. – Introduction.

As the energy in pion nucleon collisions increases beyond the $\frac{3}{2}$, $\frac{3}{2}$ resonance we enter the region where inelastic processes become important. Of these the most striking are the reactions in which K-mesons and hyperons are produced. In the following lectures we shall restrict the discussion to these phenomena. This neglect of the elastic scattering of pions or the inelastic scattering leading to additional pions is not meant to imply that these processes are unimportant but rather that at present we are not ready to interpret the information they reveal.

Of couse, in our knowledge of the strange particles and their interactions we are also very much near the beginning. Nevertheless, in this initial period of exploration, of particle identification, lifetime measurement, attempted spin determination and rough cross section measurement, a considerable degree of success has been achieved in noticing important regularities in the complex pattern. All of our present evidence concerning the strange particles fits naturally into a schema of classification originally proposed by Gell-Mann [1] and by Nishijima [2]. For a deeper understanding of the nature of the classification scheme we shall have to wait for a comprehensive theory of the fundamental particles. However, in providing a framework for experimental results and for suggesting experiments the Gell-Mann-Nishijima scheme has already proved of very great value [3].

Our first task will be to show how this classification arises if one adopts a certain point of view toward the experimental evidence.

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2. – Classification of strange particles according to the scheme of Gell-Mann and Nishijima.

Let us begin by listing the well-established particles together with their masses and approximate life-times [4]:

Particle		Mass (MeV)	Mean life (s)
Photon	γ	0	stable
Leptons	ν	0	stable
	e^-	0.510976	stable
	μ^-	(105.70 ± 0.06)	$(2.22 \pm 0.02) \cdot 10^{-6}$
Mesons	$\pi^\pm$	(139.63 ± 0.06)	$(2.56 \pm 0.05) \cdot 10^{-8}$
	π^0	(135.04 ± 0.16)	$< 4 \cdot 10^{-16}$
	$K^\pm$	(494.0 ± 0.2)	$(1.224 \pm 0.013) \cdot 10^{-8}$
	K^0	(494.4 ± 1.8)	K_1^0: $(0.95 \pm 0.08) \cdot 10^{-10}$ K_2^0: $(4 < \tau < 13) \cdot 10^{-8}$
Baryons	p	(938.213 ± 0.01)	stable
	n	(939.506 ± 0.01)	$(1.04 \pm 0.13) \cdot 10^{3}$
	Λ^0	(1115.2 ± 0.14)	$(2.77 \pm 0.15) \cdot 10^{-10}$
	Σ^+	(1189.4 ± 0.25)	$(0.83^{+0.06}_{-0.05}) \cdot 10^{-10}$
	Σ^-	(1196.5 ± 0.5)	$(1.67 \pm 0.17) \cdot 10^{-10}$
	Σ^0	$(1190.5^{+0.9}_{-1.4})$	theoretically $\sim 10^{-19}$
	Ξ^-	(1320.4 ± 2.2)	$(4.6 < \tau < 200) \cdot 10^{-10}$

More graphically we may plot a mass spectrum of these particles, separating them according to charge (drawing not to scale):

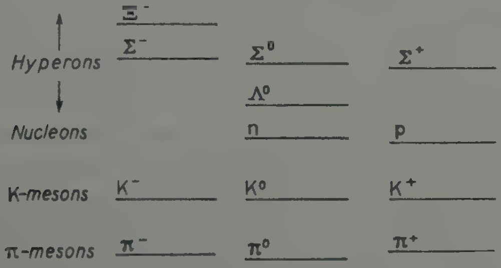

Fig. 1.

Most of these particles were first discovered in the cosmic radiation where they were identified by their decay modes. For the hyperons:

$$\begin{aligned}
\Lambda^0 &\to p + \pi^- + \ 37\ \text{MeV}\,, && (\sim 3\cdot 10^{-10}\ \text{s}),\\
&\to n + \pi^0 + \ 41\ \text{MeV}\,, &&\\
\Sigma^+ &\to p + \pi^0 + 116\ \text{MeV}\,, && (\sim 1\cdot 10^{-10}\ \text{s}),\\
&\to n + \pi^+ + 110\ \text{MeV}\,, &&\\
\Sigma^0 &\to \Lambda^0 + \gamma + \ 75\ \text{MeV}\,, && (\text{very fast}),\\
\Sigma^- &\to n + \pi^- + 117\ \text{MeV}\,, && (\sim 2\cdot 10^{-10}\ \text{s}),\\
\Xi^- &\to \Lambda^0 + \pi^- + \ 66\ \text{MeV}\,.
\end{aligned}$$

For K-mesons the principal decay modes are:

$$\begin{aligned}
K^+ &\to \pi^+ + \pi^+ + \pi^- + \ 75\ \text{MeV}\,, && (\sim 10^{-8}\ \text{s}),\\
&\to \pi^+ + \pi^0 \qquad + 219\ \text{MeV}\,, &&\\
&\to \pi^+ + \pi^0 + \pi^0 + \ 84\ \text{MeV}\,, &&\\
&\to \pi^0 + e^+ + \nu \ + 358\ \text{MeV}\,, &&\\
&\to \pi^0 + \mu^+ + \nu \ + 253\ \text{MeV}\,, &&\\
K_1^0 &\to \pi^+ + \pi^- + \qquad 215\ \text{MeV}\,, && (\sim 10^{-10}\ \text{s}),\\
&\to \pi^0 + \pi^0 + \qquad 224\ \text{MeV}\,, &&\\
K_2^0 &\to e^- + \pi^+ + \nu\,, && (\sim \text{few}\cdot 10^{-8}\ \text{s}),\\
&\to \mu^- + \pi^+ + \nu\,, &&\\
&\to \ldots
\end{aligned}$$

All these lifetimes are very long on a nuclear scale where we might expect them to be of order

$$\hbar/Mc^2\,,$$

if the interaction responsible for the decay were strong. Instead, an estimate of the effective coupling strength in the decay interactions shows that they are comparable in magnitude to the weakest interactions we know, those responsible for nuclear β decay, μ-meson decay and pion decay. Moreover it has recently been discovered that the decay of the hyperons and K-mesons, in common with other weak interactions share the property of non-conservation of parity. However the fact that these strange particles are found in cosmic radiation in relative abundance and are readily produced in high collisions of pions and nucleons with nuclei, shows that the interactions responsible for the production are strong, and comparable to the pion-nucleon and nucleon-nucleon interactions. This produces the dilemma that any strong interaction of pions with nucleons to create a strange particle along with other pions and nucleons in the final state, must also be responsible for a very rapid decay of the strange particle. For example, suppose there were a strong production reaction which

$$\pi^- + p \rightarrow \Lambda^0 + \pi^0 ,$$

might proceed schematically according to the Feynman diagram shown in Fig. 2. We should then also have a very strong decay process

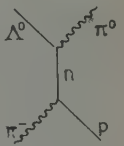

Fig. 2.

$$\Lambda^0 \rightarrow \pi^- + p ,$$

which might proceed according to the diagram of Fig. 3.

The way out of this dilemma was revealed by the detailed experimental investigations of the production process, mainly in the collision of negative pions with protons and was in fact first anticipated theoretically by several physicists. We now have good experimental evidence that the production mechanism requires always at least *two* strange particles in the final state produced in *association* with one another. Thus there are many examples of the reactions

$$\begin{aligned} \pi^- + p &\rightarrow \Lambda^0 + K^0 , \\ &\rightarrow \Sigma^- + K^+ , \\ &\rightarrow \Sigma^0 + K^0 , \end{aligned}$$

where bubble chamber photographs allow clear identification of the final particles from their decay modes. There is no production event recorded which is inconsistent with this *law of associated production.*

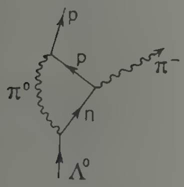

Fig. 3.

In the classification scheme of Gell-Mann and Nishijima this fact of associated production is interpreted in terms of a selection rule which may be extracted in a natural way from the experimental evidence. The motivation comes from the observation that the strange particles appear to be grouped in charge multiplets as we see by examining the masses of K^+, K^-, K^0 and of Σ^+, Σ^0, Σ^-. In common with the pion isotopic triplet and the nucleon isotopic doublet it is therefore assumed that one can meaningfully attach a total isotopic spin I to each multiplet of hyperons or K-mesons. The sepa-

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rate members of the multiplet are identified according to their charge Q by a relation of the form

$$Q = I_3 + \text{const}\,,$$

where I_3 ranges in integral steps over the $2I+1$ values from $-I$ to $+I$. For example for pions

$$Q = I_3\,, \qquad I_3 = 1, 0, -1\,, \qquad \text{for } \pi^+, \pi^0, \pi^-,$$

and for nucleons

$$Q = I_3 + \tfrac{1}{2}\,, \qquad I_3 = \tfrac{1}{2}, -\tfrac{1}{2}\,, \qquad \text{for p, n.}$$

For the strange particles the constant to be added to I_3 will play the central role in the discussion, as we shall see.

Extrapolating from the experience with the pion-nucleon interaction it is assumed that in the strong interactions of pions, nucleons and strange particles, the total isotopic spin is conserved. This implies, of course, that in strong interactions each component of the total isotopic spin is conserved. The component I_3, in particular, plays a special role because of its connection with the electric charge Q, which is also conserved. Thus from conservation of the total Q and the total I_3 it follows that the sum of the « constant » terms in the relation $Q = I_3 + \text{const}$ for each of the interacting particles is likewise conserved. Let us write the « constant » in the form $\frac{1}{2}Y$; that is

$$Q = I_3 + \tfrac{1}{2}Y\,.$$

Since Q is an integer, Y will likewise be an integer, even or odd for a given multiplet according to whether I is an integer or half integer. Thus for pions and nucleons:

$$Y = 0\,, \qquad \text{for } \pi^+, \pi^0, \pi^-,$$

$$Y = 1\,, \qquad \text{for p, n.}$$

At this point it is important to emphasize that we have reasons to expect that the quantity Y is conserved not only in the strong interactions but in electromagnetic interactions as well. The electromagnetic interactions do not respect charge independence. They are presumably responsible for the small mass differences between $\pi^{\pm}$ and π^0, n and p, and between the Σ^+, Σ^0 and Σ^-. However it can be shown that the electromagnetic interactions will nevertheless conserve the component I_3 if it is assumed that the fundamental electromagnetic coupling is through the charge. In that case, since the charge is a function of I_3, the electromagnetic coupling will transform in isotopic spin space like the third component of isotopic spin, and will thus commute with the total I_3. The conservation of Y then, follows from the conservation of Q.

To have some indication of what the quantity Y might signify it is instructive to consider its value for the anti-proton ($\overline{\mathrm{p}}$) anti-neutron ($\overline{\mathrm{n}}$) doublet. From the possibility of the strong interaction

$$\mathrm{p}+\overline{\mathrm{p}} \to \pi^0 + \pi^0 ,$$

it follows that p and $\overline{\mathrm{p}}$ must have opposite values of I_3, $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively. This implies

$$Y = -1 , \qquad \text{for } \overline{\mathrm{p}}, \overline{\mathrm{n}}.$$

The change in Y in going from the nucleons to the anti-nucleons suggests that we make explicit use of the *law of conservation of nucleons*. The stability of the proton and of nuclei in general show the existence of an absolute selection rule which forbids transformations in which a single nucleon changes, for example, into a meson or photon and an electron. There is now experimental evidence for the rule that a nucleon can only be created or destroyed together with a corresponding anti-nucleon. Since nucleons and hyperons are coupled by the weak decay interactions we must extend the conservation law to include these particles as well. Introducing the generic term baryon for nucleon or hyperon, the law of baryon conservation may be expressed as follows. Let B denote the number of baryons minus the number of anti-baryons; then B is rigorously conserved by all known interactions, strong or weak.

For the pions and nucleons it is consistent to write

$$Y = B , \qquad \text{for pions and nucleons.}$$

That this cannot be true for the strange particles as well is evident from the example of the Λ^0 where $Y=0$, and $B=1$, assuming that the Λ^0 is an isotopic singlet with $I=0$. To take account of the difference let us write

$$Y = B + S ,$$

or

$$Q = I_3 + \tfrac{1}{2}B + \tfrac{1}{2}S ,$$

where S is an integer which is zero for pions and nucleons and as we shall see *different* from zero for the strange particles. For this reason, S is called the *strangeness* quantum number. In virtue of the separate conservation of Q, I_3, and B we see that S is rigorously conserved by all strong or electromagnetic interactions. Conservation of strangeness is in general violated only by the weak decay interactions.

3. – Assignment of strangeness and isotopic spin to the particles.

Let us now see how one can assign isotopic spins and strangeness quantum numbers to the hyperons and K-mesons which are known.

3·1. *The* Λ^0 *singlet.* – The Λ^0 is evidently an isotopic singlet; no charged particles with mass close to that of the Λ^0 have been observed. Since Q and I_3 vanish, and $B=1$, we conclude that $S=-1$. For energetic reasons it is impossible for the Λ^0 to decay into a nucleon and a K-meson in a way which could conserve strangeness. Hence the Λ^0 is long-lived; the slow decay into $p+\pi^-$ involves a change of strangeness by one unit: $\Delta S=1$.

3·2. *The* Σ *triplet.* – The identification of Σ^+, Σ^0 and Σ^- among the hyperons indicates an isotopic spin $I=1$ for this multiplet. Q is therefore to be equated with I_3, and since $B=1$, we conclude that $S=-1$, as for the Λ^0. The Σ^+ and Σ^- do not have enough energy to decay into $n+K$ or $\Lambda^0+\pi$ and are therefore metastable. The Σ^0 on the other hand undergoes a rapid electromagnetic decay,

$$\Sigma^0 \to \Lambda^0 + \gamma\,,$$

which conserves the strangeness. A number of Σ^0 decays of this type have been observed, for example, by the Columbia [5] group, where the Λ^0 was identified by its decay and the γ identified by its conversion in the propane of a liquid propane bubble chamber.

Another fast reaction

$$\Sigma^- + p \to \Lambda^0 + n\,,$$

observed in the liquid hydrogen bubble chamber at Berkeley [6] is likewise consistent with the conservation of strangeness.

Leaving the Ξ^- temperarily let us consider next the

3·3. K-*mesons.* – From the existence of two charged, and a neutral K-meson we might be tempted to assign isotopic spin $I=1$. Since $B=0$, this would imply $S=0$, and would allow the *rapid* decay of the K-meson into pions. In fact the life-time against pion decay is very long.

Evidence for a different assignment of strangeness comes from the detection of the reaction (fast)

$$\pi^- + p \to \Sigma^- + K^+\,,$$

in bubble chambers [7]. From the assignment $S = -1$ for the Σ^- we must have: $S = +1$ for the K^+. With baryon number $B = 0$ this implies $I_3 = +\frac{1}{2}$ for K^+. The other member of this multiplet with $I_3 = -\frac{1}{2}$ would then be the K^0.

From this point of view the K^- would not be a member of the same isotopic spin multiplet as the K^+. Instead the K^- might naturally be regarded as the anti-particle to the K^+ with *opposite* strangeness $S = -1$ and opposite $I_3 = -\frac{1}{2}$. (In general particle and anti-particle should have opposite strangeness since they can presumably be created together in a strangeness conserving interaction of γ-rays with nucleons.) As the other partner along with K^- in the $S = -1$ isotopic doublet we would have to assume that there existed a particle $\overline{K}^0$, the antiparticle to K^0.

In fact the experimental evidence indicates a vast qualitative difference in the behaviour of K^- and K^+ which strongly supports these choices of strangeness quantum numbers.

Thus, in contrast to the many examples of associated Σ^-, K^+ production in $\pi + p$ collisions (113 examples in the Columbia experiments) no case has been seen of the reaction

$$\pi^- + p \to \Sigma^+ + K^-,$$

which would be forbidden according to the strangeness selection rule.

Since there are no hyperons of positive strangeness, the K^+ (with $S = -1$) interactions with protons are limited to elastic scattering

$$K^+ + p \to K^+ + p,$$

or possibly scattering with pion production if the energy is high enough. For the K^-, on the other hand, the existence of hyperons of negative strangeness permits a great variety of reactions in addition to scattering. For example, in the Berkeley liquid hydrogen bubble chamber the following reactions have been seen:

$$\begin{aligned} K^- + p &\to K^- + p, \\ &\to \overline{K}^0 + n, \\ &\to \Sigma^+ + \pi^-, \\ &\to \Sigma^- + \pi^+, \\ &\to \Sigma^0 + \pi^0, \\ &\to \Lambda^0 + \pi^0. \end{aligned}$$

These are all consistent with the choice $S = -1$ for K^-.

Additional support for the choice of opposite strangeness for K^+ and K^- comes from the fact that at the Brookhaven cosmotron and also at the Berkeley bevatron with protons of 3 and 6 GeV respectively there is a marked preponderance of K^+ production over K^- production in collisions of protons with the target. This may be interpreted by noticing that the lowest energy threshold for the production of K^- is obtained with the reaction

$$p+p \rightarrow p+p+K^-+K^+ .$$

This reaction has a threshold of 2.5 GeV. For the K^+ production the existence of hyperons of strangeness -1 makes possible the reaction

$$p+p \rightarrow \Lambda^0+p+K^+ ,$$

which has a threshold of only 1.5 GeV. The large difference in threshold is readily observable in the excitation curves and easily accounts for the large excess of K^+ over K^-.

Finally, there is now good experimental evidence that the K^0 and $\overline{K}^0$ are different states, but a complete discussion of this question is too complicated to undertake here.

Prior to the discovery that parity was not conserved in the weak interactions there were thought to exist two distinct types of K-mesons, for example τ^+ and θ^+, characterized by the decay modes [8]

$$\tau^+ \rightarrow \pi^+ + \pi^+ + \pi^- ,$$

$$\theta^+ \rightarrow \pi^+ + \pi^0 .$$

Although τ^+ and θ^+ were found to have essentially the same mass and lifetime within the experimental errors, a careful analysis of the energy distribution of the three pions from the τ^+ indicated that the final pion states of τ^+ and θ^+ must have opposite parity. As is known now, this feature is a manifestation of the general property of all known weak interactions to contradict the principle of reflection symmetry. There is therefore no reason to distinguish any longer between the τ and θ but instead they should be regarded as different decay modes of the same particle.

3·4. *Cascade hyperons.* – The negative hyperon Ξ^- which undergoes slow decay to $\Lambda^0+\pi^-$ has as yet no certain strangeness assignment. It could be a singlet with strangeness -3 or a member of a doublet with strangeness -2, in which case a Ξ^0 should also exist. These choices are forced by the fact that no positive counterpart or doubly charged negative counterpart exists as far as we know. There may be some preference for $S = -2$ but the situation is left open pending more definite experimental information on the production and interactions of this particle. (*)

3·5. – To summarize the situation, we have made the following assignments:

		S	I
Hyperons:	Λ^0	-1	0
	Σ^+, Σ^0, Σ^-	-1	1
	Ξ^-, (Ξ^0)	-2	$\frac{1}{2}$
K-mesons:	K^+, K^0	$+1$	$\frac{1}{2}$
	K^-, $\bar{K}^0$	-1	$\frac{1}{2}$

From the law that strangeness is conserved in strong and electromagnetic interactions one finds simultaneously an explanation for the experimental fact of associated production of strange particles from initial states of vanishing strangeness and for the fact that the strange particles are long lived or metastable from the point of view of the nuclear time scale. All the experimental information available is in support of the Gell-Mann-Nishijima classification.

It should be remembered that the primary emphasis so far has been on the conservation law for strangeness but that additional restrictions of a more quantitative nature can be derived from the requirement of isotopic spin conservation. Just as in pion physics there are important equalities and inequalities between cross sections which will probably soon be tested experimentally.

As an illustration, in the interaction of K-mesons with deuterium (isotopic spin zero) we have the relations between cross sections [9]:

$$d\sigma(K^- + d \to \Lambda^0 + n + \pi^0) = \tfrac{1}{2}\, d\sigma(K^- + d \to \Lambda^0 + p + \pi^-)\,,$$

and

$$d\sigma(K^- + d \to \Sigma^0 + \pi^- + p) = d\sigma(K^- + d \to \Sigma^+ + \pi^0 + p)\,.$$

(*) *Note added in proof.* - Production of the neutral cascade hyperon, Ξ^0, in the reaction

$$K^- + p \to K^0 + \Xi^0\,,$$

has recently been reported. L. W. Alvarez, P. Eberhard, M. L. Good, W. Graziano, H. K. Ticho and S. G. Wojcicki: *Phys. Rev. Lett.*, **2**, 215 (1959).

As an example of the type of inequalities that must be satisfied consider the reactions:

$$\pi^- + p \rightarrow \Sigma^- + K^+ ,$$

$$\pi^- + p \rightarrow \Sigma^0 + K^0 ,$$

$$\pi^+ + p \rightarrow \Sigma^+ + K^+ :$$

From the hypothesis that there are two reaction amplitudes, one for isotopic spin $\frac{3}{2}$ and the other for isotopic spin $\frac{1}{2}$, which are sufficient to describe these reactions, one can deduce triangle inequalities of the same type as for the pion-nucleon scattering:

$$(a) \qquad \sqrt{2\sigma(\Sigma^0)} \leqq \sqrt{\sigma(\Sigma^-)} + \sqrt{\sigma(\Sigma^+)} ,$$

$$(b) \qquad \sqrt{\sigma(\Sigma^-)} \leqq \sqrt{2\sigma(\Sigma^0)} + \sqrt{\sigma(\Sigma^+)} ,$$

$$(c) \qquad \sqrt{\sigma(\Sigma^+)} \leqq \sqrt{2\sigma(\Sigma^0)} + \sqrt{\sigma(\Sigma^-)} .$$

These relations should hold for differential as well as total cross sections.

The present data from the Columbia bubble chamber indicate that these inequalities are satisfied within the experimental accuracy.

4. - Production of strange particles in pion-proton collisions.

Most of the detailed information that is at present available on the production of strange particles comes from the experiments on pion-proton collisions in the energy range around 1000 MeV. The observations have been made with liquid propane and liquid hydrogen bubble chambers using the pion beams from the Brookhaven cosmotron and the Berkeley bevatron. Although the statistical accuracy can be very much improved, and will be as these beautiful experiments continue, they have already given very significant results:

(i) the trend of the differential cross sections and the variation of the total cross sections with energy have been clarified;

(ii) the experiments on the decay of the hyperons produced in the π-p collisions give definite proof that parity is not conserved in these weak decay processes;

(iii) there is convincing evidence, but not quite a proof, that the Λ^0 and Σ^- hyperons have spin $\frac{1}{2}$.

4·1. – *Differential and total cross sections for strange particle production.* – We list the reactions studied and the approximate number of events observed, following the recent summary report of STEINBERGER to the CERN Conference (1958).

Reaction	No. of events
$\pi^- + p \to \Lambda^0 + K^0$	Approx. 1000
$\pi^- + p \to \Sigma^0 + K^0$	150
$\pi^- + p \to \Sigma^- + K^+$	250
$\pi^+ + p \to \Sigma^+ + K^+$	32

In most of the experiments a magnetic field of 13.4 kG was used, thus providing a means for measuring the momenta of the products from the hyperon and K-meson decays.

For the total cross sections the results may be summarized by saying that in the energy range 1000 to 1300 MeV the cross sections are of the order of several tenths of a millibarn. The Λ^0 cross section shows a well-defined maximum in the neighborhood of 1000 MeV pion kinetic energy. The other cross sections vary more smoothly with energy above the threshold.

The angular distributions show large departures from isotropy as the energy increases above threshold. The Λ^0 distribution shows a strong backward maximum in the center of mass system, while the Σ^- and Σ^+ distributions are peaked forward. The Σ^0 distribution does not so far show any special features.

4·2. – *Demonstration of parity non-conservation in hyperon decay.* – Following the experimental discovery that parity was not conserved in the weak interactions responsible for nuclear β-decay, $\pi \to \mu$ decay, and $\mu \to e$ decay, it was natural to enquire whether parity conservation was also violated in the weak hyperon decays and in the decays of K-mesons.

The original stimulation to Lee and Yang came from the apparent failure of parity conservation in the K^+ decay into two or three pions, which almost certainly represented states of different parity. Nevertheless there was until recently no obvious experimental asymmetry which could provide clear and immediate evidence for a parity violation as was obtained in the case of the other weak interactions.

Such evidence has recently been obtained by the experimental groups at Brookhaven and at Berkeley [10] following a suggestion of LEE, STEINBERGER *et al.* [11].

If parity were not conserved in the decay of the Λ^0 hyperon, for example, this would most readily reveal itself in the angular distribution of the decay products from a *polarized* hyperon, as in the corresponding case for the decay from a polarized nucleus. Thus we might expect a forward-backward asymmetry of pion emission in the decay $\Lambda^0 \to p + \pi^-$, relative to the spin direction of the hyperon. For producing a hyperon in a state of known polarization it was suggested to use the production reaction

$$\pi^- + p \to \Lambda^0 + K^0 ,$$

where general arguments based on reflection invariance in *strong* interactions show that the Λ^0 will in general be polarized, to a larger or smaller extent, normal to the plane of the outgoing particles (production plane). The magnitude of the hyperon polarization, P, will depend on the productionangle ω (shown for the collision, as drawn in Fig. 4, in the center of mass system).

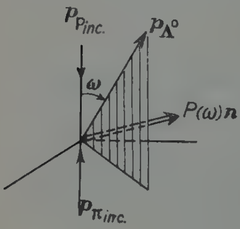

Fig. 4.

If parity is not conserved in the hyperon decay, and if it is assumed that the hyperon has spin $\frac{1}{2}$, the distribution in the decay angle θ of the pion relative to the polarization axis in the hyperon rest frame must have the form

$$I(\theta) = 1 + P(\omega)\alpha \cos\theta ,$$

where α is the anisotropy coefficient for completely polarized hyperons. The experimental result for the Λ^0 shows conclusively that α does not vanish.

The most direct way to extract the information concerning α from the experimental observations of decay angles θ is to observe that if one includes all production angles ω and takes the average value of $\cos\theta$, then

$$\langle \cos\theta \rangle = \tfrac{1}{3}\,\overline{P}\alpha ,$$

where $\overline{P}$ is the average of the polarization $P(\omega)$, averaged over the production angles. From all the data in the energy range 910 MeV to 1100 MeV available at the time of the quoted articles, one finds for the Λ^0

$$\langle \cos\theta \rangle = 0.17 \pm 0.03 ,$$

or

$$\overline{P}\alpha = 0.52 \pm 0.10 ,$$

where the positive sign means that the pions are emitted preferentially in the direction $\boldsymbol{p}_{\pi\,\text{inc}} \times \boldsymbol{p}_{\Lambda^0}$.

Measurements of a similar type were also made for the Σ^- using the reactions

$$\pi^- + p \to \Sigma^- + K^+ ,$$

and

$$\Sigma^- \to \pi^- + \mathrm{n}\,.$$

In the case of the Σ^- decay no statistically significant anisotropy was observed. This could mean either a small value of α or a small average polarization $\overline{P}$ (or both) for the Σ^-.

To find the value of α from these data it is necessary to know the average polarization $\overline{P}$ of the Λ^0, obviously very difficult to obtain at present. Taking the spin of the Λ^0 to be $\frac{1}{2}$, the Columbia group has suceeded, however, in setting a lower limit for $|\alpha|$ on the assumption that in the Λ^0 production only s and p wave amplitudes enter, a reasonable hypothesis in the energy range near threshold. In this case, as for $\pi+\mathrm{p}$ scattering, the differential cross section and the polarization of Λ^0 are expressible in terms of three complex amplitudes a, b and c by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |a + b\cos\omega|^2 + |c|^2 \sin^2\omega\,,$$

$$P(\omega) = 2\,\frac{\sin\omega}{\mathrm{d}\sigma/\mathrm{d}\Omega}\,\mathrm{Im}\,c^*(a + b\cos\omega)\,.$$

With the addition of the asymmetry parameter α it should then be possible to describe the Λ^0 production and decay in detail. Investigation of this problem shows that the data do not permit such a representation unless $|\alpha|$ is larger than 0.78 approximately.

$$|\alpha| \gtrsim 0.78\,.$$

As a by-product to this analysis it was found that the different possible outgoing angular momentum states for the $\Lambda^0+\mathrm{K}^0$ enter with comparable strength in the angular distribution within rather wide limits.

Recently Lee and Yang [12], and Gatto [13], have discussed an independent mean of measuring the asymmetry parameter α. Consider the decay of a hyperon in its rest frame, assuming its spin to be $\frac{1}{2}$. If χ_i is the initial spinor for the hyperon, the decay may be described by giving the form of the spinor χ_f for the outgoing wave of final nucleon and pion. Allowing parity not to be conserved, the final state will be a combination of $s_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ which, for symmetry reasons, can be written in the general form

$$\chi_f = (A + B\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})\chi_i\,,$$

where $\hat{\boldsymbol{p}}$ is a unit vector in the direction of the outgoing pion, and A and B are complex amplitudes. Parity non conservation is indicated here by the presence of the pseudoscalar quantity $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}$. In this method of presentation the final nucleon is treated non-relativistically, but the extension to the relativistic case is not difficult and does not change the results.

The angular distribution of the decay products will be proportional to the norm of the outgoing spinor. Thus

$$\chi_f^\dagger \chi_f = \chi_i^\dagger (A^* + B^*\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})(A + B\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})\chi_i = \{|A|^2 + |B|^2\}\left\{1 + \frac{A^*B + AB^*}{|A|^2 + |B|^2}(\chi_i^\dagger\boldsymbol{\sigma}\chi_i)\cdot\hat{\boldsymbol{p}}\right\}.$$

The average of $\chi_i^\dagger\boldsymbol{\sigma}\chi_i$ over the initial hyperon spin states may be identified with its initial polarization. Comparing with our previous expression, we see that the parameter α is

$$\alpha = \frac{2\,\mathrm{Re}\,A^*B}{|A|^2 + |B|^2}.$$

Now, as a consequence of the parity violation in the decay one can expect that the outgoing nucleon from the hyperon decay will exhibit a net polarization even if the hyperons are initially unpolarized. This polarization will of course be longitudinal. To calculate the polarization, or equivalently, to find $\langle\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\rangle$ it is convenient to regard the initial hyperon state as an incoherent mixture of the two states χ_+ and χ_- with spin parallel and opposite to $\hat{\boldsymbol{p}}$ respectively.

Then:

$$\langle\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\rangle = \frac{\chi_+^\dagger(A^* + B^*\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}(A + B\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})\chi_+ + \chi_-^\dagger(A^* + B^*\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}(A + B\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})\chi_-}{\chi_+^\dagger(A^* + B^*\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})(A + B\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})\chi_+ + \chi_-^\dagger(A^* + B^*\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})(A + B\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})\chi_-} =$$

$$= \frac{(A^* + B^*)(A + B) - (A^* - B^*)(A - B)}{(A^* + B^*)(A + B) + (A^* - B^*)(A - B)} = \frac{AB^* + BA^*}{|A|^2 + |B|^2}.$$

Therefore:

$$\langle\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\rangle = \alpha\,.$$

Consequently a measurement of the nucleon polarization will allow a determination of the magnitude and sign of α. In practice the nucleon will exhibit a *transverse* polarization component in the laboratory system in virtue of the motion of the hyperon, acquired in the production process. Subsequent scattering of the nucleon may then reveal its polarization. An experiment of this type may be feasible. Indeed preliminary attempts are under way.

It is important to notice that if α can be determined with some accuracy, it will be eventually possible to use the measurements on production and decay of the hyperon as a means for finding the hyperon polarization.

4·3. – *Determination of hyperon spin.* – LEE and YANG [14] have also pointed out that the observed strong asymmetry in the decay of the polarized Λ^0 may be used to put limits on its possible spin. On the basis of angular momentum conservation alone they have succeeded in obtaining restrictions on the angular distribution of the decay products in terms of inequalities depending on the spin J. For example if θ is the angle between the decay proton and the hyperon polarization in the hyperon rest frame, the average value of $\cos\theta$ over any sample of hyperons is shown to lie between the limits

$$-\frac{1}{2J+2} \leqslant \langle\cos\theta\rangle \leqslant \frac{1}{2J+2}\,.$$

In the case $J=\frac{1}{2}$ the upper limit $\frac{1}{3}$ for $|\langle\cos\theta\rangle|$ can be easily understood from our previous expression for the decay angular distribution since the polarization $P(\omega)$ and the asymmetry parameter α cannot exceed unity in absolute value.

The present experimental value of 0.17 for $\langle\cos\theta\rangle$ is consistent with a spin of $\frac{1}{2}$ and just consistent with spin $\frac{3}{2}$. In fact, sharper and more complicated restrictions on the angular distribution show already a preference for spin $\frac{1}{2}$. One can hope that additional data in the near future will unambiguously settle the question. (*)

Another method for finding hyperon spins through a study of their decay angular distribution has been suggested by ADAIR [15]. This method rests on the assumption that the K-mesons have spin zero, an hypothesis which is not proved but is strongly suggested by a variety of experimental results. With the independent procedure of Lee and Yang for obtaining the hyperon spin alone the application of Adair's method will yield information on the K-meson spin.

The argument proceeds as follows. In the production process

$$\pi + \mathrm{p} \to \mathrm{Y}\ \text{(hyperon)} + \mathrm{K}\,,$$

consider only those outgoing waves with vanishing component of orbital angular momentum m in the direction of the incident pion (z-axis). If the K has spin zero, the z-component of angular momentum of the hyperon (Y), m_{Y}, will coincide with that of the initial proton. Thus, whatever the intrinsic spin of the hyperon, it will under these conditions emerge with $m_{\mathrm{Y}}=\frac{1}{2}$ or

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(*) *Note added in proof.* - Recent application of the method of Lee and Yang to a sample of 614 decays shows definitely that the Λ^0 spin is $\frac{1}{2}$: F. S. CRAWFORD, M. CRESTI, M. L. GOOD, M. L. STEVENSON and H. L. TICHO: *Phys. Rev. Lett.*, 2, 114 (1959).

$m_Y = -\frac{1}{2}$ with the two states in general equally populated. In the subsequent decay of the hyperon

$$Y \to \pi + \text{Nucleon}\,,$$

there will occur amplitudes of opposite parity which are capable of interfering with one another, as we have seen. However, if one includes together decays with the same absolute value of $\cos\theta$, where θ is the angle of the decay in the hyperon rest frame measured from the z-direction, the interference term is eliminated. The resulting distribution in $|\cos\theta|$ is then uniquely determined by the hyperon spin.

If the spin is $\frac{1}{2}$ it is evident from our previous discussion that the symmetrized distribution will be independent of $|\cos\theta|$. For spin $\frac{3}{2}$ and $\frac{5}{2}$, for example, one finds the distribution $F(|\cos\theta|)$ given by

Spin J	$F(\lvert\cos\theta\rvert)$
$\frac{1}{2}$	1
$\frac{3}{2}$	$\frac{1}{2}(1+3\cos^2\theta)$
$\frac{5}{2}$	$\frac{3}{4}(1-2\cos^2\theta+5\cos^4\theta)$

These expressions are easily found by writing the wave function for the final $\pi+$ Nucleon system as a linear combination of the functions $F^m_{l,j}$ introduced earlier, for $l=j+\frac{1}{2}$ and $l=j-\frac{1}{2}$ with $m=\frac{1}{2}$ or $m=-\frac{1}{2}$.

Measurement of $F(|\cos\theta|)$ for these hyperons produced in the state $m_L=0$ should therefore fix the hyperon spin. The condition $m_L=0$ is realized for hyperons produced exactly in the forward or exactly in the backward direction with respect to the incident pion. In practice one can estimate from the observed angular distribution of the production process how far one may go from the forward or backward directions and still avoid a contamination of $m_L \neq 0$ greater than some given small amount.

This method of analysis has been applied by the combined groups at Columbia, Brookhaven, Michigan, Bologna, and Pisa [16] to their bubble chamber observations of the processes

$$\pi^- + p \to \Lambda^0 + K^0\,,$$

and

$$\pi^- + p \to \Sigma^- + K^+\,,$$

at pion laboratory kinetic energies in the range $(900 \div 1300)$ MeV. For both hyperons the decay angular distributions are consistent with isotropy and in definite contradiction with the spin $\frac{3}{2}$ or spin $\frac{5}{2}$ distribution.

REFERENCES

[1] M. Gell-Mann: *Phys. Rev.*, **92**, 833 (1953).

[2] T. Nakano and K. Nishijima: *Progr. Theor. Phys. Japan*, **10**, 581 (1953).

[3] For excellent reviews of the field see: R. H. Dalitz: *Rep. Progr. Phys.*, **20**, 163 (1957) and M. Gell-Mann and A. H. Rosenfeld: *Ann. Rev. Nucl. Sci.*, **7**, 407 (1957).

[4] Table taken from: W. H. Barkas and A. H. Rosenfeld: Radiation Laboratory Report, UCRL-8030 (March 1958).

[5] R. Plano, N. Samios, M. Schwartz and J. Steinberger: *Nuovo Cimento*, **5**, 216 (1957).

[6] L. W. Alvarez, H. Bradner *et al.*: UCRL-3583 (1956) and UCRL-3775 (1957).

[7] F. Eisler, R. Plano *et al.*: Columbia University Report Nevis, 70 (1958).

[8] R. H. Dalitz: *Phil. Mag.*, **44**, 1068 (1953); F. Fabri: *Nuovo Cimento*, **11**, 479 (1954),

[9] M. Gell-Mann: *Suppl. Nuovo Cimento*, **4**, 848 (1956); T. D. Lee: *Phys. Rev.*, **99**, 337 (1955).

[10] F. Eisler, R. Plano, A. Prodell, N. Samios, M. Schwartz, J. Steinberger, P. Bassi, V. Borelli, G. Puppi, H. Tanaka, P. Waloschek, V. Zoboli, M. Conversi, P. Franzini, I. Mannelli, R. Santangelo, A. Silvestrini, D. A. Glaser, C. Graves and M. L. Perl: *Phys. Rev.*, **108**, 1353 (1957); F. S. Crawford, M. Cresti, M. L. Good, K. Gottstein, E. M. Lyman, F. T. Solmitz, M. L. Stevenson and H. K. Ticho: *Phys. Rev.*, **108**, 1102 (1957).

[11] T. D. Lee, J. Steinberger, G. Feinberg, P. K. Kabir and C. N. Yang: *Phys. Rev.*, **106**, 1367 (1957).

[12] T. D. Lee and C. N. Yang: *Phys. Rev.*, **108**, 1645 (1957).

[13] R. Gatto: *Nucl. Phys.*, **5**, 183 (1958).

[14] T. D. Lee and C. N. Yang: *Phys. Rev.*, **109**, 1755 (1958).

[15] R. Adair: *Phys. Rev.*, **100**, 1540 (1955).

[16] F. Eisler, R. Plano, A. Prodell, N. Samios, M. Schwartz, J. Steinberger, P. Bassi, V. Borelli, G. Puppi, H. Tanaka, P. Waloschek, V. Zoboli, M. Conversi, P. Franzini, I. Mannelli, R. Santangelo, V. Silvestrini, G. L. Brown, D. A. Glaser and C. Graves: *Nuovo Cimento*, **7**, 222 (1958).

SUPPLEMENTO AL VOLUME XIV, SERIE X
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4° Trimestre

Weak Interactions of Fermions.

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1. – Introduction.

We are going to restrict the discussion to the three fermion interactions which occur in the original form of the Puppi triangle. The Puppi triangle says that the members of any two pairs of particles can be exchanged simultaneously by means of the Fermi interaction. There have been attempts to add other legs corresponding to the strange particles, but the situation is not entirely clear yet so we will omit any reference to them here.

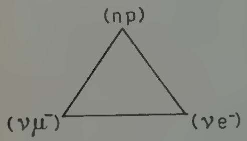


Fig. 1.

Considerable progress has recently been made in the study of two of these three legs and the final shape of the interaction responsible has turned out to be rather simple. Rather than take the experiments and try to deduce the interaction (which is not uniquely fixed) it is clearer to take an assumed interaction and show how it explains the experimental results. The assumed interaction has the merit of being very simple, and therefore desirable.

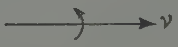
We will make the following assumptions. (It is taken for granted that the number of baryons is conserved).

1) The number of leptons minus the number of antileptons is conserved.

The leptons are: e^-, ν, μ^-,

The antileptons are: e^+, $\tilde{\nu}$, μ^+.

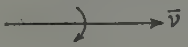
2) The neutrino is always left handed (has negative helicity):



i.e.

$$\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p} = -1\,.$$

The anti-neutrino is always right handed:



3) The coupling is of the form $(V\text{-}A)$ with the same coupling constant, g, in all three legs. This is called the « universal Fermi interaction ». For many years there were considerable obstacles in the way of a such a universal interaction, but these were dramatically removed last year.

We shall need some simple theory to make our qualitative statements. We will use four-vectors:

$$x_\mu = (x_1, x_2, x_3, it)\,. \qquad \text{using } \hbar = c = 1\,.$$

(Greek suffixes represent 4-vectors)

$$p_\nu = (p_1, p_2, p_3, iE) = -i\frac{\partial}{\partial x_\nu}\,.$$

In particular $p_4 = iE = -i(\partial/\partial x_4)$. The Dirac equation can be written:

$$\gamma_\mu \frac{\partial\psi}{\partial x_\mu} + M\psi = 0\,.$$

(The repeated suffix μ indicates summation over all values). In the Dirac equation we must satisfy $p^2 + M^2 = E^2$ and this implies that,

(1) $$\gamma_\lambda\gamma_\mu + \gamma_\mu\gamma_\lambda = 2\delta_{\mu\lambda}\,.$$

We define

(2) $$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4\,.$$

From the relations (1) we can easily show that

(3) $$\gamma_5^2 = 1 \qquad \text{and} \qquad \gamma_\lambda\gamma_5 + \gamma_5\gamma_\lambda = 0\,,$$

i.e. γ_5 has the same properties as γ_μ. We next define some spin operators:

(4) $$\sigma_k = i\gamma_4\gamma_5\gamma_k\,.$$

(The roman suffix $k = 1, 2, 3$).

Let us now verify that $\boldsymbol{\sigma}\cdot\boldsymbol{p}$ does produce the required results when we operate with it on a spinor wave function.

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\varphi = i\lambda_4\gamma_5\gamma_k\left[-i\,\frac{\partial\varphi}{\partial x_k}\right] = \gamma_4\gamma_5\gamma_k\,\frac{\partial\varphi}{\partial x_k}\,.$$

Now $\gamma_k(\partial\varphi/\partial x_k)$ appears in the Dirac equation from which we may therefore write

$$\gamma_k\,\frac{\partial\varphi}{\partial x_k} = -M\varphi - \gamma_4\,\frac{\partial\varphi}{\partial x_4}\,,$$

$$\therefore\;\; \boldsymbol{\sigma}\cdot\boldsymbol{p}\varphi = \gamma_4\gamma_5\left[-M - \gamma_4\,\frac{\partial}{\partial x_4}\right]\varphi\,.$$

sing the properties of the γ matrices we get

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\varphi = \gamma_5(-E + \gamma_4 M)\varphi\,,$$

i.e.

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 &= \gamma_5(-E+\gamma_4 M)\,\gamma_5(-E+\gamma_4 M) = \\ &= (-E-\gamma_4 M)(-E+\gamma_4 M) = \\ &= (E^2 - \gamma_4^2 M^2) = \\ &= E^2 - M^2 = \\ &= p^2\,,\end{aligned}$$

$$\therefore\;\; \left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p}\right)^2 = 1\,.$$

Thus $(\boldsymbol{\sigma}\cdot\boldsymbol{p})/|p|$ has eigenvalues ±1.

Let us call the two eigenfunctions: φ_R and φ_L, then

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\boldsymbol{p})\varphi_R &= \;\;|p|\varphi_R\,,\\ (\boldsymbol{\sigma}\cdot\boldsymbol{p})\varphi_L &= -|p|\varphi_L\,.\end{aligned}$$

Consider φ_R:

$$|p|\varphi_R = (\boldsymbol{\sigma}\cdot\boldsymbol{p})\varphi_R = \gamma_5(-E + k_4 M)\varphi_R\,.$$

Going to the extreme relativistic limit:

$$\frac{M}{E}\to 0 \qquad \text{and} \qquad E\to p\,.$$

Further for positive energy particles $|p| = E$, and so,

$$E\varphi_R = -E\gamma_5\varphi_R .$$

So that,

$$\varphi_5 = -\gamma_5\varphi_R ,$$

and similarly

$$\varphi_L = \gamma_5\varphi_L .$$

In the non-relativistical case, a factor of v/c has to be introduced in the left hand side of the equations.

We now arrive at the relativistic relations

$$\begin{cases} \frac{1}{2}(1+\gamma_5)\varphi_R = 0 , \\ \frac{1}{2}(1-\gamma_5)\varphi_R = \varphi_R ; \end{cases}$$

$$\begin{cases} \frac{1}{2}(1-\gamma_5)\varphi_L = 0 , \\ \frac{1}{2}(1+\gamma_5)\varphi_L = \varphi_L . \end{cases}$$

Thus whenever the term $(1 \pm \gamma_5)\varphi$ appears in an equation and φ refers to a relativistic positive energy particle we know that it is left handed (+) or right handed (—).

2. – The interaction.

In Dirac theory the quantity

$$\varphi^+\gamma_4\gamma_\mu\varphi ,$$

represents a four-component current vector. Therefore if we contract this with another similar vector we obtain a scalar which is therefore suitable for describing the interaction between particles in a Lorentz invariant way. We then obtain:

$$g(\varphi_p^+\gamma_4\gamma_\mu\varphi_n)(\varphi_e^+\gamma_4\gamma_\mu\varphi_\nu) + \text{h.c.} .$$

The Hermitian conjugate produces the reverse reaction. When the Dirac equation undergoes second quantization the φ's and φ^+'s become operators which respectively destroy and create a particle of the kind described by the field. Thus in the above equation if φ_n describes a neutron field then this operator destroys a neutron and φ_p will create a proton.

The above matrix element therefore destroys a neutron and creates a proton, and also destroys a neutrino and creates an e^-,

$$n+\nu \rightarrow p+e^-,$$

or

$$n \rightarrow p+e^-+\bar{\nu},$$

since the destruction of a particle is the same thing as the creation of its antiparticle.

This form of coupling is known as the vector (V) type as it is formed by the contraction of two quantities which have vector properties under Lorentz transformation. As the contraction of two vectors produces a scalar, the interaction is scalar.

There are actually five kinds of interaction which can be written down in terms of the γ matrices, but only one other now appears to be important in the weak Fermi Interactions. This is the axial vector (A) which is obtained by the contraction of two quantities which behave like axial vectors under transformation. Once again the resultant interaction is a scalar. The operator in this case is obtained from our σ matrices.

$$g(\varphi_p^+ i\gamma_4\gamma_5\gamma_\mu\varphi_n)(\varphi_{e^-}^+ i\gamma_4\gamma_5\gamma_\mu\varphi_\nu) + \text{h.c.}\,.$$

We have used the same coupling constant g but have inserted a minus sign in agreement with the assumption of (V-A) interaction.

The (V-A) interaction can now be written as the sum of these two. However we have not yet satisfied the assumption that the neutrino is left handed. Therefore we amend the interaction by inserting the appropriate handedness operator, $\frac{1}{2}(1+\gamma_5)$ in front of the neutrino field. It is convenient also to multiply by 2 to define the new g. Thus we obtain:

$$(V\text{-}A) \rightarrow 2g(\varphi_p^+\gamma_4\gamma_\mu\varphi_n)(\varphi_{e^-}^+\gamma_4\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\varphi_\nu) - \\ - 2g(\varphi_p^+ i\gamma_4\gamma_5\gamma_\mu\varphi_n)(\varphi_{e^-}^+ i\gamma_4\gamma_5\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\varphi_\nu) + \text{h.c.}\,.$$

The introduction of the γ_5 has not affected the behaviour of the interaction under rotation. It has, however, introduced a quantity which changes sign under reflection (the handedness), and so the reaction does not conserve parity. This is obvious because the L.H. neutrino is not changeable into a R.H. neutrino by simple reflection, since one of the basic assumptions is that the R.H. neutrino does not exist.

The axial vector expression may be rewritten:

$$+\,2g\{\psi_{\rm p}^{+}\gamma_4\gamma_\mu\gamma_5\psi_{\rm n}\}\{\psi_{\rm e^-}^{+}\gamma_4\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\psi_\nu\}+\text{h.c.}\,.$$

Here the factors i have been multiplied out, $\gamma_5\gamma_\mu$ has been replaced by $-\gamma_\mu\gamma_5$ in both brackets, and $\gamma_5(1+\gamma_5)$ has been put equal to $1+\gamma_5$. Thus the V-A interaction is of the form:

$$AC+BC+\text{h.c.}=(A+B)C+\text{h.c.}\,,$$

where

$$A=2g(\psi_{\rm p}^{+}\gamma_4\gamma_\mu\psi_{\rm n})\,,$$

$$B=2g(\psi_{\rm p}^{+}\gamma_4\gamma_\mu\gamma_5\psi_{\rm n})\,,$$

$$C=(\psi_{\rm e^-}^{+}\gamma_4\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\psi_\nu)\,.$$

The final form of the interaction is therefore:

1) β decay:

$$4g(\psi_{\rm p}^{+}\gamma_4\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\psi_{\rm n})(\psi_{\rm e^-}^{+}\gamma_4\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\psi_\nu)+\text{h.c.}\,, \tag{5}$$

and for the other two legs of the triangle.

2) μ decay:

$$4g(\psi_\nu^{+}\gamma_4\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\psi_{\mu^-})(\psi_{\rm e^-}^{+}\gamma_4\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\psi_\nu)+\text{h.c.}\,.$$

3) μ capture:

$$4g(\psi_{\rm n}^{+}\gamma_4\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\psi_{\rm p})(\psi_\nu^{+}\gamma_4\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\psi_{\mu^-})+\text{h.c.}\,.$$

We shall consider now the physical processes brought about by these interactions.

2·1. β *decay*. – *a*) Measurements of the lifetimes of β-emitters (in particular ^{14}O and the free neutron) indicate that:

$$g\mu_\pi^2\sim 1.6\cdot 10^{-7},$$

μ_π is the π-meson mass, and is introduced to give g the correct dimensions ($\hbar=c=1$).

b) The electrons emitted should be predominantly left-handed (negative helicity) and positrons predominantly right-handed (positive helicity). This may be seen by moving the projection operator for left-handedness; $(1+\gamma_5)$,

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through the second bracket in expression (5) to act on the electron creation operator $\psi_{e^-}^+$, γ_5 commuting with $\gamma_4\gamma_\mu$.

The handedness of the positrons has been established experimentally by scattering, and by other methods, and agrees with the above assignment.

c) It is predicted that the strength of the Fermi coupling constant is equa to that of the Gamow-Teller, *i.e.* that the vector and axial vector interaction are present in equal proportions.

The nuclear matrix elements are of the form:

Fermi: $\left|\int \psi_f^+ \psi_i \, d\tau\right|^2$,

Gamow-Teller: $\left|\int \psi_f^+ \boldsymbol{\sigma} \psi_i \, d\tau\right|^2$.

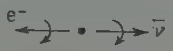
^{14}O is an example of a pure Fermi transition; ^6He is an example of a pur Gamow Teller transition. It should be noted that on the average Gamow Teller matrix elements are three times larger than Fermi matrix element because of the three possible spin orientations.

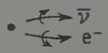
Lifetime measurements indicate that:

$$g^2_{\text{G.T.}} = 1.25 g^2_{\text{F}} .$$

The difference may be due to some mesonic interaction between the nucleons

d) We can determine the angular correlation of the e⁻ and the $\bar{\nu}$. Fo a vector interaction ($\Delta J = 0$) the e⁻ and $\bar{\nu}$, whose helicities are already deter mined, travel in predominantly the same direction to conserve angular mo mentum. For an axial vector interaction ($\Delta J = \pm 1$) they travel in predomi nantly opposite directions. Thus,

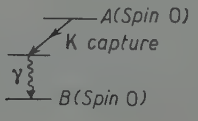
Axial Vector (Gamow-Teller): 

Vector (Fermi): 

The ^6He experiment used to contradict this, but has since been repeated an agrees now with the theory.

e) GOLDHABER measured the handedness of the neutrino directly.

He used the following scheme:



$$A + e_K^- \rightarrow \overset{\uparrow}{B^*} + \nu \qquad \text{excited state}$$

$$B^* \rightarrow B + \gamma$$

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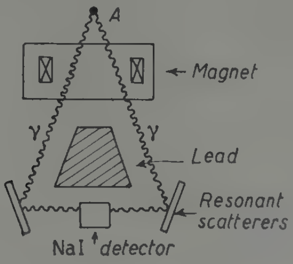
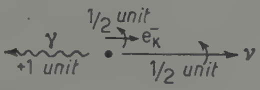
Nucleus A decays by K capture to an excited state of nucleus B. Nucleus B then passes to its ground state emitting a photon. The spins of nucleus A and the ground state of nucleus B are both zero.

The apparatus used is shown schematically below:



Photons of maximum energy are resonant scattered at a target made of « B », *i.e.* the photons are resonant scattered into the detector provided that there is no Doppler shift introduced by nuclear recoil. If thisis so, than we have the following situation:

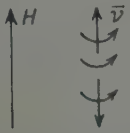


As $\Delta J = 0$ and because the photon carries unit angular momentum, a left-handed neutrino must give rise to a right-handed photon. The handedness (circular polarization) of the photon was measured by the sign of the difference in transmission through a magnet on reversing the field — the direction of magnetization being approximately in the direction of transmission. The experiment showed that neutrinos were left handed.

f) In β-decay of aligned ^{60}Co nuclei one expects a backward asymmetry.

$$^{60}\text{Co} \rightarrow {}^{60}\text{Ni} + e^- + \bar{\nu}\,. \qquad \Delta J = -1 \;\; (5 \rightarrow 4)\,.$$

That this is so, may be seen from the following diagram:



R.H. ½ unit
1 unit carried away in transition
e^- L.H. ½ unit

This was observed by Wu *et al.*

g) An experiment by Telegdi *et al.* on the electron-neutrino correlation on the decay of the free neutron establishes that the interaction is V-A.

There is no known experimental fact which contradicts this form of the interaction.

If the interaction is universal, we may consider it fixed for the other two legs of the triangle, and hence for the interaction:

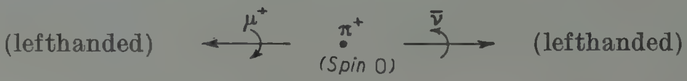
$$\pi^+ \to \mu^+ + \nu\,,$$

$$(\text{Actually } \pi^+ \to p + n \to \mu^+ + \nu)\,,$$

we expect the following; using leg 3.

1) Lifetime $\sim 10^{-8}$ s. (This is verified experimentally).

2) μ^+ fully polarized and left handed. This may be seen directly from the following diagram.



Also the μ^- is fully polarized and right handed. It should be noted that only axial vector or pseudo-scalar interaction in leg 3 allow this interaction to occur. The handedness of the muon has not been measured as yet, but no doubt will be in the future.

2·2. μ *decay* (Leg (2))

$$\mu^+ \to e^+ + \nu + \bar{\nu}\,.$$

The theory allows us to predict the following:

1) The lifetime. Using the value g (the V coupling constant obtained from β decay) the theory gives $\tau = 2.2 \cdot 10^{-6}$ s. The value is in fact within 2% of the measured value. As the prediction is precise we have now to decide whether or not this 2% is accidental, since the β decay should be affected by virtual pion effects.

To avoid this deliberately requires a modification of β decay and should produce effects which are observable by other means.

2) That the e^+ is completely right handed and the e^- is completely left handed. These predictions follow from manipulation of the $(1 \pm \gamma_5)$ terms in the matrix element for muon decay.

The first verification of this result was carried out by Holt *et al.* at Liverpool and later and more accurately by Crowe at Berkeley.

3) The distribution of positrons with energy and angle to the positive muon spin is,

$$\mathrm{d}N(x,\theta) \sim \left\{(3-2x)+\cos\theta\,(2x-1)\right\} x^2\,\mathrm{d}x\,\mathrm{d}\Omega\,,$$

where $x=$ electron energy/maximum electron energy ,

$$\theta = \frac{\boldsymbol{\sigma}_\mu\cdot\boldsymbol{p}_e}{p_e} = \varphi - \pi\,,$$

and

$$\varphi = \frac{\boldsymbol{p}_e\cdot\boldsymbol{p}_\mu}{p_e p_\mu} \quad \text{(because the } \mu^+ \text{ is left handed)}\,.$$

Now for e^+:

$$\int \mathrm{d}N(x,\theta)\,\mathrm{d}x = 1+\frac{1}{3}\cos\theta = 1-\frac{1}{3}\cos\varphi\,,$$

i.e. the positron asymmetry parameter is $-\frac{1}{3}$. The current experimental value is (-0.29 ± 0.04). There are a number of reasons why the experimental value might be low. The geometrical factors which have to be considered in such a measurement are uncertain as the muon source is diffused.

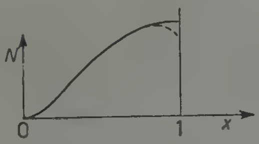
The integral

$$\int \mathrm{d}N(x,\theta)\,\mathrm{d}\Omega \sim (3-2x)x^2\,\mathrm{d}x\,.$$

Comparison with the expression for $\mathrm{d}N(x)$ in terms of the Michel ϱ parameter shows that the appropriate value is

$$\varrho = 0.75\,.$$

This gives a curve of the form:



However the experimental value obtained by Crowe is

$$\varrho = (0.68\pm0.02)\,,$$

and this gives a modification shown by the dotted line. The radiative correction has already been applied to obtain this value of 0.68, however recalculation shows that it may be possible to raise this correction to give $\varrho = 0.70$.

A possible explanation of the discrepancy which still exists is that the interaction is non-local: the greater the non-locality, the lower the value of the ϱ parameter predicted by theory.

The expression for $\mathrm{d}N(x, \theta)$ also gives the result:

$$\frac{\text{high energy electron asymmetry}}{\text{average asymmetry}} = 3 .$$

This is confirmed by several experiments.

2·3. β *decay of the pion* (Leg (1))

$$\pi^+ \to e + \nu .$$

This is obtained from the Puppi triangle by the following chain:

$$\pi^+ \to (p + \bar{n}) \xrightarrow[(1)]{} e^+ + \nu .$$

The branching ratio (with respect to $\pi^+ \to \mu^+ + \nu$) predicted by the theory is $1.3 \cdot 10^{-4}$ whereas the lowest published result is that of Anderson and Lattes which is $< 2 \cdot 10^{-5}$ with 99% confidence. A number of other experiments are at present in progress but as yet there is neither confirmation nor contradiction of the above limit.

2·4. *Radiative* β *decay of the pion.*

$$\pi^+ \to e^+ + \nu + \gamma .$$

The (V-A) theory predicts a branching ratio $\sim 10^{-7}$ which comes almost entirely from the V part of the interaction. This is consistent with the experimental limit of less than $7 \cdot 10^{-6}$.

If the interaction contained a tensor component as indicated by the old β experiments, the branching ratio would be $\sim 10^{-4}$. This would then be inconsistent with the value of $5 \cdot 10^{-6}$ the experiments indicate for such an interaction. (The same experimental results indicate $< 7 \cdot 10^{-6}$ for vector interaction and $< 5 \cdot 10^{-6}$ for tensor due to a difference in angular correlations).

2·5. *Muon capture* (Leg (3))

$$\mu^- + p \to n + \nu .$$

The theoretical predictions here are at present in advance of experimental results.

1) Life times are at least of the correct magnitude and are consistent with a universal coupling constant.

2) Selection rules afford little information on the coupling as the neutrino has a high energy and therefore the expansion into multipoles is less useful. Consequently the final states are not restricted and the calculation of the matrix elements cannot be performed with certainty. The present indications are towards a greater Gamow-Teller than Fermi components.

3) The neutron asymmetry and neutron polarization are both able to give information on the coupling present. The former is probably just within the scope of present techniques but no results are yet available. The latter appears to be too difficult to measure at present.

4) Capture of μ^- to form a β unstable nucleus.

Determination of the subsequent electron distribution allows the muon spin direction to be evaluated. No results have yet been obtained.

The present situation in muon capture is therefore uncertain and there is no definite evidence on the nature of the interaction in leg (3), apart from the information already noted under pion decay.

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Present Problems in the Theory of Weak Interactions.

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In these lectures I would like to review some still unsettled problems in the theory of weak interactions and present some related speculations. After the brilliant confirmation of most of the predictions of the universal AV theory of Feynmann, Gell-Mann, Marshak and Sudarshan (F.G.M.S. theory) it seems rather safe to base the discussion on the hypotheses of that theory. We shall not review here the development of the F.G.M.S. theory, in accordance with our program. We shall try however to make these lectures essentially self-contained so that no preliminary reading of other papers on the subject will be necessary. Finally we want to warn the reader that most of the suggestions and points of view that will be expressed here are mostly based on speculations which have little experimental basis. It should not be surprising if some of them will eventually turn out to be wrong.

1. – Let me remind you of two different points of view that have been proposed to describe weak interactions. The older description of weak interactions consisted in grouping the particles into pairs, such as ($\mu\nu$), (eν), (pn) etc., and assuming that each pair interacts with the other pairs. A different description can now be proposed in the AV theory. With axial and vector couplings there is a possibility that the weak interaction lagrangian can be written as a coupling of a current J_μ with itself, $J_\mu J_\mu^\dagger$. Of course J_μ is not a current in the usual sense, since both vector and axial contribution are present. One difference with respect to the older description is, for instance, that now there are couplings also of the different pairs to themselves, such as (pn)(pn). Moreover the new description is certainly much more specific. According to which of the two descriptions we want to assume we can either classify the different pairs into a few classes (lepton class, $\mathcal{L}$, and various strongion (*) classes $\mathcal{S}_0$, $\mathcal{S}_{-1}$, $\mathcal{S}_1$)

(*) By strongion we mean a strong interacting particle: baryon or a meson (wit the exception of the μ-meson which is a lepton).

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or, correspondingly, decompose the current J_μ into different components. For brevity we shall denote with the same symbol of the class the corresponding current, so that

$$J_\mu = \mathcal{L}_\mu + \mathcal{S}_{0\mu} + \mathcal{S}_{-1\mu} + \mathcal{S}_{1\mu}\,. \tag{1}$$

We shall examine in detail the various groups.

$\mathcal{L}$, lepton class (the corresponding current $\mathcal{L}_\mu$ is the lepton current). It consists of the pairs

$$(\mu^+\bar{\nu})\,, \qquad (e^+\bar{\nu})\,. \tag{2}$$

For a pair $(A\bar{B})$ the transfer of charge is defined by $\Delta Q = Q_B - Q_A$. It must be noted that other possible pairs such as:

$$(\mu\bar{e})\,, \quad (\nu\bar{\nu})\,, \quad (\mu\bar{\mu})\,, \quad (e\bar{e}) \tag{3}$$

are excluded from $\mathcal{L}$. Their inclusion would lead to processes such as

$$\mu^+ \to e^+ + e^- + e^+\,, \qquad \mu^- + p \to e^- + p \tag{4}$$

not observed. We also note that the excluded pairs have $\Delta Q = 2$ or $\Delta Q = 0$. Therefore it seems that one has to assume at this point that only pairs with $\Delta Q = \pm 1$ are coupled. We shall come back later in more detail to this point. To be definite, without limitation to our discussion, let us consider here only the pairs with $\Delta Q = -1$. Thus the lepton current is given by $\mathcal{L}_\mu = (\mu^+\nu)_\mu + (e^+\bar{\nu})_\mu$.

$\mathcal{S}_0$: strongion class with $\Delta S = 0$ (the corresponding current is the strangeness conserving current). One finds a total of six possible baryon pairs which satisfy the limitations $\Delta Q = -1$, $\Delta S = 0$ (and of course also $\Delta N = 0$ because they must couple to the lepton pairs):

$$(p\bar{n})\,, \quad (\Sigma^+\bar{\Lambda})\,, \quad (\Sigma^+\bar{\Sigma}^0)\,, \quad (\Lambda\bar{\Sigma}^-)\,, \quad (\Sigma^0\bar{\Sigma}^-)\,, \quad (\Xi^0\bar{\Xi}^-)\,. \tag{5}$$

All such pairs can go into one another by strong interaction. (We shall call such pairs « equivalent »). They are also equivalent to the π^+. It will be convenient to add to these pairs possible boson terms corresponding to vector interactions of the boson currents $i(\varphi(\partial\varphi^*/\partial x_\mu) - \varphi^*(\partial\varphi/\partial x_\mu))$ with the lepton fields. One finds two such terms

$$(\pi^+\pi^0)\,, \qquad (K^+K^0)\,. \tag{6}$$

The presence of such terms is necessary if one accepts the hypothesis of nonrenormalizability for the vector part, as we shall see later. Finally we note that linear boson terms of the form

$$\frac{\partial\varphi}{\partial x_\mu} \tag{7}$$

could also be included in the axial strangeness conserving current (here φ is the pion field). Such terms would be required as counter terms to have a renormalizable theory with weak interactions only up to the first order. On the other hand a recent calculation of the $\pi \to \mu + \nu$ rate by GOLDBERGER and TREIMAN does not include such terms and leads to a surprisingly good agreement with experiment.

$\mathcal{S}_{-1}$, strongion class with $\Delta S = -1$ (strangeness non-conserving current of the first type). We find a total of six pairs

$$(p\bar{\Lambda}), \quad (p\bar{\Sigma}^0) \quad (n\bar{\Sigma}^-), \quad (\Lambda\bar{\Xi}^-), \quad (\Sigma^+\bar{\Xi}^0), \quad (\Sigma^0\bar{\Xi}^-) . \tag{8}$$

They are equivalent to the K^+. Two bilinear meson terms are possible, namely

$$(K^+\pi^0), \qquad (K^0\pi^+) . \tag{9}$$

Differently from the corresponding situation with $\mathcal{S}_0$, there is nothing here forcing the assumption that (9) are coupled. The only observation that can be made is that the simultaneous existence of both $K \to \pi + \mu + \nu$ and $K \to \mu + \nu$ excludes the possibility that in $\mathcal{S}_{-1}$ only the boson terms (9) are coupled. This can be shown readily by parity considerations. Also nothing can be said about linear boson terms of the form (8), where φ is the K field.

By coupling to each other $\mathcal{L}$, $\mathcal{S}_0$ and $\mathcal{S}_{-1}$ one can account for all the observed weak processes. The requirement that the pairs carry a unit charge (or equivalently, that the current J_μ is singly charged) has eliminated unwanted processes such as

$$\mu^+ \to e^+ + e^- + e^+, \quad \mu^- + p \to e^- + p, \quad K \to \mu^\pm + e^\mp + \pi, \quad K \to \pi + \nu + \bar{\nu} . \tag{10}$$

By classifying the pairs into classes one avoids the embarassing question of which pairs in each class are originally coupled: for instance only one pair in each class may be coupled, the other pairs will then still be indirectly coupled through the strong interactions, or all pairs may be coupled according to some particular unknown symmetry. It is certainly relevant to investigate the possibilities for such symmetries. We shall touch upon this question when speaking of the possible non-renormalization of the vector part.

2. – The two strongion classes $\mathcal{S}_0$ and $\mathcal{S}_{-1}$ introduced so far can be characterized by saying that $\mathcal{S}_0$ contains all pairs equivalent to the π^+, and $\mathcal{S}_{-1}$ all pairs equivalent to the K^+. Moreover, for $\mathcal{S}_{-1}$; $\Delta S = \Delta Q$.

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We shall now examine at limitations arise from whthe suggested selection rule $\Delta S = \pm 1$ for strange particle decays. Let us in fact consider the class $\mathcal{S}_{+1}$.

$\mathcal{S}_{+1}$, strongion class with $\Delta S = \pm 1$ (strangeness non-conserving current of the second kind). It contains

(11) $$(\Sigma^+ \overline{n}) , \qquad (\Xi^0 \Sigma^-) .$$

and a possible bilinear meson term

(12) $$(\overline{K}^0 \pi^+) .$$

There is no linear boson term possible for such a class. Pairs of this class would be equivalent to a positive meson with $S = -1$. No such a meson exists in the Gell-Mann scheme. The system $(\overline{K}^0\pi^+)$ instead behaves like a positive meson with $S = -1$. Now the point is that if we couple $\mathcal{S}_{+1}$ to $\mathcal{S}_{-1}$ we find a possible reaction such as $\Xi^0 \to n + \Sigma^- + \Sigma^-$ which would lead to the sequence

(13) $$\Xi^- \to \Xi^0 + \pi^- \to n + \overline{\Sigma}^- + \Sigma^- \to n + \pi^- .$$

Now $\Xi^- \to n + \pi^-$ has not been seen and instead $\Xi^- \to \Lambda^0 + \pi^-$ has been seen in all the cases reported (not so many however) thus suggesting the selection rule $\Delta S = \pm 1$ for strange particle decays [2]. This is a reasonably strong motivation for excluding class $\mathcal{S}_{+1}$: What is important is that one can then obtain a number of predictions which can be compared with experiment. Let us first see what these predictions are. Then we shall look into the question of how one might exclude this class by theoretical arguments. By coupling $\mathcal{S}_{+1}$ to $\mathcal{L}$ one would have the decay modes

(14) $$\Sigma^+ \to n + l^+ + \nu , \qquad \Xi^0 \to \Sigma^- + l^+ + \nu ,$$

(where l means either e or μ), and

(15) $$\overline{K}^0 \to \pi^- + l^+ + \nu ,$$

and its charge conjugate

(15′) $$K^0 \to \pi^+ + l^- + \overline{\nu} .$$

Such decay modes are forbidden if $\mathcal{S}_{+1}$ is not coupled. What does this mean in terms of the physical particles K_1^0 and K_2^0? Recalling that [3]

(16) $$K_1^0 = \frac{1}{\sqrt{2}} (K^0 + \overline{K}^0) , \qquad K_2^0 = \frac{1}{\sqrt{2}} (K^0 - \overline{K}^0) ,$$

one finds immediately the following relations between the decay amplitudes

$$(K_1^0 \to l^+) = (K_2^0 \to l^+) = \frac{1}{\sqrt{2}} (K^0 \to l^+)\,, \tag{17}$$

$$(K_1^0 \to l^-) = -(K_2^0 \to l^-) = -\frac{1}{\sqrt{2}} (\overline{K}^0 \to l^-)\,. \tag{17'}$$

In (17), (17′), $(K_1^0 \to l^+)$, for instance, is the matrix element for $K_1^0 \to l^+ + \pi^- + \nu$.

Okun and Pontecorvo have pointed out that the presence or absence of $\Delta S = \pm 2$ interactions is of great relevance in the problem of the K_1^0, K_2^0 mass difference [4]. If $\mathcal{S}_{+1}$ is coupled, a term contributing to the mass difference would already be present at the first order in the weak coupling constant, for instance, through the virtual scheme

$$\overline{K}^0 \to \Sigma^+ + \overline{n} + \pi^- \to \overline{\Xi}^0 + \Sigma^+ + \pi^- \to K^0\,. \tag{18}$$

The Pais-Piccioni oscillations [5] would then be very frequent ($\sim 10^{17}$ per second) as compared to the inverse lifetime ($\sim 10^{10}\ s^{-1}$). If $\mathcal{S}_{+1}$ is not coupled, a typical mass difference contribution would come from

$$\overline{K}^0 \to p + \overline{n} + \pi^- \to K^0\,, \tag{19}$$

with two weak steps, leading to a mass difference of the same order of the inverse lifetime. In this situation one may observe the Pais-Piccioni oscillations.

The requirement $\Delta S = \Delta Q$ excludes class $\mathcal{S}_{+1}$. A stronger requirement which also leads to exclude class $\mathcal{S}_{+1}$ is that the strangeness non-conserving current transforms in isotopic spin space like the component of a spinor. One sees immediately that there is no way of satisfying this requirement in the class $\mathcal{S}_{+1}$ since there are no suitable linear combinations available, for instance of a Σ and a nucleon. We shall later discuss again such question.

Finally, if one still wants to complicate the situation, one can think of currents $\mathcal{S}_{\pm 2}$ for which $\Delta S = 2$. Again one meets the trouble of $\Xi^- \to n + \pi^-$ by coupling for instance a term $(n\Xi^-)$ to $\mathcal{S}_0$. Also decay modes such as $\Xi^- \to n + e^- + \overline{\nu}$ would be expected in that case.

3. – Let us now discuss the possibility of neutral currents ($\Delta Q = 0$). We define two neutral lepton currents, $\mathcal{L}^{(0)}$, containing $(\mu^+\mu^-)$, (e^+e^-) and $(\nu\nu)$, and $\mathcal{L}^{(0)'}$ containing (μ^+e^-). $\mathcal{L}^{(0)'}$ is apparently not coupled at all. One finds out that is it still consistent with present experiments to couple $\mathcal{L}^{(0)}$ to a strangeness-conserving neutral current $\mathcal{S}_0^{(0)}$.

$\mathcal{S}_0^{(0)}$: neutral strongion class with $\Delta S = 0$ (neutral strangeness-conserving current). It may contain the baryon terms

$$(p\bar{p}),\ (nn),\ (\Lambda^0\Lambda^0),\ (\Sigma^+\Sigma^+),\ (\Sigma^0\Sigma^0),\ (\Sigma^0\Lambda^0),\ (\Sigma^-\Sigma^-),\ (\Xi^0\Xi^0),\ (\Xi^-\Xi^-)$$

and the meson terms

$$(\pi^+\pi^-),\quad (\pi^0\pi^0),\quad (K^+K^-),\quad (K^0\bar{K}^0)\,. \tag{21}$$

The coupling $\mathcal{L}^{(0)}\mathcal{S}_0^{(0)}$ does not lead to any inconsistency with present data. There would be weak interactions such as $\mu + n \to \mu + n$ which however will be hard to detect because of the stronger electromagnetic effects, and processes such as $\nu + p \to \nu + p$ also hard to detect. One also will have interactions $(e^+e^-)(\nu\bar{\nu})$ and $(\mu^+\mu^-)(\nu\nu)$ which can be rearranged to look as $(e\bar{\nu})(e\bar{\nu})$ and $(\mu\bar{\nu})(\mu\bar{\nu})$ again in the AV combination, and they would therefore interfere with the corresponding terms arising directly from the charged currents. A similar conclusion also holds for terms $(pp)(nn)$ which can be rearranged in the form $(pn)(p\bar{n})$.

The reason which suggested that we look into the possibility of a neutral current is connected with an extension of the hypothesis of non-renormalization of the vector part of the strangeness-conserving $\mathcal{S}_0$, which we shall discuss later. From such a point of view, in trying to build up the neutral current $\mathcal{S}_0^{(0)}$ one is strongly tempted to require that its vector part be the 3 current, $t_\mu^{(3)}$, of the isotopic spin, the axial part being then fixed by the requirement that the fermions occur in the $\frac{1}{2}(1+\gamma_5)$ projection. This would lead automatically to the non-renormalization for the vector coupling constant in $\mathcal{S}_0^{(0)}$, and would produce a highly symmetric scheme. In such a case $\mathcal{S}_0^{(0)}$ would consist of

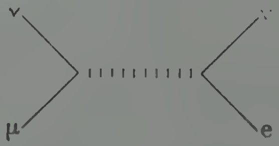
$$(p\bar{p}),\ (n\bar{n}),\ (\Sigma^+\Sigma^+),\ (\Sigma^-\Sigma^-),\ (\Xi^0\Xi^0),\ (\Xi^-\Xi^-),\ (\pi^+\pi^-),\ (K^+K^-),\ (K^0\bar{K}^0)\,. \tag{22}$$

One has further to look into the possibility of neutral strangeness non-conserving currents. One is here however immediately faced by a difficulty. In fact, by coupling a term $(n\Lambda)$ with $(\nu\nu)$, (e^+e^-), and $(\mu^+\mu^-)$, one would find processes such as $K^0 \to e^+ + e^-$ and $K^0 \to \mu^+ + \mu^-$ of which the first one would proceed at a very low rate, but the second one could very well be observed. Even more, one would find processes such as $K^- \to \pi^- + \nu + \nu$ and $K^- \to \pi^- + e^+ + e^-$, which would favorably compete with $K^- \to \pi^0 + e^- + \nu$, which is a known reaction.

4. – If one wants to keep only the charged currents, a rather suggestive hypothesis is that weak interactions are mediated by an agent X (for instance a massive boson) which carries unit charge. Each pair would interact with X, and indirectly with all the other pairs coupled to X. The resulting interaction between two pairs would then be slightly non-local. If X is a boson with mass M, it must have spin 1 to obtain in the local limit the AV theory, and its propagator would be in momentum space

$$\frac{1}{K^2+M^2}\left(\delta_{\mu\nu}+\frac{K_\mu K_\nu}{M^2}\right). \tag{23}$$

The non locality contained in the $\delta_{\mu\nu}$ terms is difficult to observe because of the smallness of the momentum transfers K^2 for most of the known processes. Even more difficult would be the observation on the $K_\mu K_\nu$ term which would contribute effective scalar and pseudoscalar contributions to the resulting direct interaction between the currents. Since X is supposed to have only electromagnetic interactions (because it is charged) and « semiweak » interactions with the currents, its production cross-section would be very small (for instance it could be produced by the reaction $\gamma\to X^+ + X^-$) so that no difficulty concerning its apparent absence may occur. Its decay would occur at first order in the semiweak interactions of X with the currents. Let us however see what difficulties one encounters by postulating the existence of X. First of all, there is evidence for some non-local effects in μ-decay which were suggested [6] to explain the deviations of the ϱ-value from $\frac{3}{4}$ (the best ϱ-value determination so far is $\varrho = 0.68 \pm 0.02$, due to CROWE). A deviation of this order of magnitude could be accounted for by values of the mass around $(300 \div 400)$ MeV. However, as we shall discuss later in more detail, the sign of the deviation cannot apparently be accounted for by a real meson. For a real meson the deviations would always be such as to increase the ϱ-value above $\frac{3}{4}$. Of course, before reaching a definitive conclusion one should try to estimate the radiative corrections to the ϱ-value for this more complicated model (in which a virtual photon can also be emitted from the non-local vertex). The radiative corrections which were substracted to get the experimental figure of 0.68 ± 0.02 are those calculated for a direct point interaction between the four fermions. A second difficulty for the hypothesis of the intermediate X occurs in connection with the apparent non-existence of the decay mode $\mu^+\to e^+ + \gamma$. If an intermediate X existed, the diagram for μ-decay, $\mu\to e+\nu+\nu$, would look as



From this diagram we can deduce the existence of an indirect term looking

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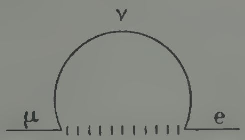
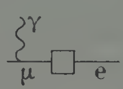
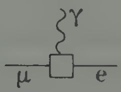
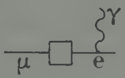
as $\mu_\rho(x)\,K_{\rho\sigma}(x-x')\,e_\sigma(x')$ in the interaction Hamiltonian, where K is some form factor. It comes from the graph



If the electromagnetic interactions are now turned on, the existence of the above term would imply the existence of diagrams of the kind







which lead to $\mu\to e+\gamma$. If the kernel K is expanded in powers of the momentum transfer and only the linear term is kept, then one can easily show that gauge invariance requires a perfect compensation of all the three graphs above so that no $\mu\to e+\gamma$ occurs at this order. This approximation is however not recommended because of the smallness of the wavelength of the emitted γ. FEYNMAN and GELL-MANN [7], and FEINBERG [8] have calculated the probability for $\mu\to e+\gamma$ with the conventional highly divergent quantum electrodynamics of a charged spin one meson X. They find a value 10^{-4} for the branching ratio of $\mu\to e+\gamma$ to $\mu\to e+\nu+\nu$. The experimental value is due to LOKANATHAN and STEINBERGER who report a figure of $2\cdot10^{-5}$ as an upper limit for the ratio. The theoretical estimate is based on a cut-off procedure and is rather uncertain. No other theoretical scheme is known however to calculate such things.

Now, why do not such difficulties occur in the local case, with a direct four fermion interaction? One easily finds that the neutrino loop (24) is zero in the local case, because of symmetry considerations. Its expression is however far from being unambiguous.

5. – Next we come to the discussion of the possible properties of the weak interactions in isotopic spin space. It may seem rather useless to start a discussion about possible properties of weak interactions in isotopic spin space, since isotopic spin is certainly non-conserved in such interactions. One may object however that, first, in a similar situation where isotopic spin is non-conserved namely with electromagnetic interactions, it is indeed very useful to analyse the amplitudes in terms of eigenstates of isotopic spin; second, there is in fact some experimental evidence for an apparently simple behaviour of the decay amplitudes in isotopic spin space, and at least the suggested rule $\Delta S=\pm1$

for strange particle decay directly implies $\Delta I_3 = \pm \frac{1}{2}$; third, some general requirements to the strangeness-conserving terms, such as that of a definite behaviour of the current [5] under the product operation of charge conjugation times charge symmetry, or that of non-renormalization for the vector part, lead us to assume a behaviour like that of the component of a vector in isotopic spin space. An important point to be noticed first is that, by requiring $\Delta I = \frac{1}{2}$ to the strangeness non-conserving current, one immediately obtains $\Delta I_3 = \pm \frac{1}{2}$ for the decay of strange particles, though in general both $\Delta I = \frac{1}{2}$ and $\Delta I = \frac{3}{2}$ will be present in the interaction. That is one has a scheme conceptually simpler than the $\Delta I = \frac{1}{2}$ rule for the interaction which still gives $\Delta I_3 = \pm \frac{1}{2}$. This result can be verified very easily. As we said already, it may very well be that $\mathcal{S}_0$, the strangeness-conserving current, behaves like a component of a vector in isotopic spin space. This follows if one postulates a definite behaviour of the current [5] with respect to the product operation of charge conjugation and charge symmetry, or also if one assumes the hypothesis of non-renormalization of the vector part. We shall discuss both these questions later. If one assumes that the strangeness non-conserving current behaves like the component of an isotopic spinor, then, by coupling it to a strangeness-conserving current behaving like a component of an isotopic vector, one would have selection rules $\Delta I = \frac{1}{2}$ and $\Delta I = \frac{3}{2}$ for the decay of strange particles. As far as the experimental situation is concerned we know that $\Delta I = \frac{1}{2}$ [9] can be at most an approximate selection rule. The main argument for it being only approximate is that one cannot explain simultaneously the ratios of the lifetime of K^0 to the lifetime of K^+ and the ratio between the probability for $K^0 \to \pi^+ + \pi^-$ and the probability for $K^0 \to \pi^0 + \pi^0$ with only $\Delta I = \frac{1}{2}$. One finds instead that one has to introduce both $\Delta I = \frac{3}{2}$ and $\Delta I = \frac{5}{2}$ contributions to account simultaneously for the above ratios. The estimated magnitude of such contributions, of the order of 10% [10] is such as to make not very plausible the suggestion that they come from electromagnetic corrections. However $\Delta I = \frac{1}{2}$ has the great merit of explaining, though not completely of course, for the above mentioned difficulty, why the K^+ has so much longer a lifetime than the K^0. If fact, if $\Delta I = \frac{1}{2}$ rigorously holds, K^+ cannot decay into two pions because, starting from $I = \frac{1}{2}$, $I_3 = \frac{1}{2}$ one can only obtain $I = 1$ for the final charged 2-meson state which should therefore be antisymmetric in space co-ordinates, in contrast with the value for the orbital angular momentum which is $l = 0$ if the K has spin zero. Moreover $\Delta I = \frac{1}{2}$ also explains the ratio of 2 of $\Lambda^0 \to p + \pi^-$ to $\Lambda^0 \to n + \pi^0$. Observation of such a value for this ratio does not however imply absence at $\Delta I = \frac{3}{2}$ contributions. It must also be noticed that $\Delta I = \frac{1}{2}$ also explains the experimental ratio $\tau'/\tau = \frac{1}{4}$. However such a value would also be obtained if a $\Delta I = \frac{3}{2}$ component were present because if the τ has spin zero the final state is expected to be predominantly symmetric in the space co-ordinates and from group theory one then can show that the possible values of the total isotopic spin of the three pions can only be $I=1$ and $I=3$. The latter, however, would not still be accessible even if $\Delta I=\frac{3}{2}$ were present. We have seen from our preceding discussion about a general description of weak interactions that it is rather difficult theoretically to present a reasonable scheme leading to $\Delta I=\frac{1}{2}$ always. The fact that $\Delta I=\frac{1}{2}$ seems to predominate in the decay of strange particles should then be explained in each case by means of a particular mechanism suppressing $\Delta I=\frac{3}{2}$ or by showing that the same ratios are also obtained by more physically plausible combinations of $\Delta I=\frac{3}{2}$ and $\Delta I=\frac{1}{2}$. An example where such a situation may occur is Λ-decay. Here one finds that by calculating directly from the coupling (pΛ)(np), keeping only the simplest diagram, one obtains the branching ratio of 2 between the decay modes $\Lambda \rightarrow p+\pi^-$ and $\Lambda \rightarrow n+\pi^0$, as with the $\Delta I=\frac{1}{2}$ rule, though the interaction now contains also a $\Delta I=\frac{3}{2}$ term.

6. – In looking for more general symmetry requirements that one woulda sk from the strongion currents, it may be convenient to recall that the strangeness-conserving current $\mathcal{S}_0$ contains all those pairs which are equivalent to a π^+. Now a π^+ is transformed into itself by performing a charge conjugation and then a charge symmetry. Therefore we can try to formulate the requirement that $\mathcal{S}_0$ has a definite behaviour under the product operation of charge conjugation and charge symmetry, and investigate to what restriction it leads for $\mathcal{S}_0$. It is unfortunate that a similar possibility does not exist for the strangeness-non-conserving currents. The current $\mathcal{S}_{-1}$ is equivalent to a K^+ and this would be transformed into an K^0 under the above product operation, so that it is impossible to formulate the problem of a definite behaviour of $\mathcal{S}_{-1}$ under this operation. A similar conclusion is also evident for $\mathcal{S}_{+1}$.

The convenience of classifying the different terms according to their properties under the product operation of charge conjugation and charge symmetry has been pointed out recently by Weinberg [11]. The relevant operator is $G=\mathcal{C}\exp[i\pi I_2]$, where $\mathcal{C}$ is the charge conjugation operator and $\exp[i\pi I_2]$ performs a rotation of 180° around the second axis in isotopic spin space, which is one way of realizing charge symmetry as a proper orthogonal transformation. For the nucleon field $G\psi G^{-1}=i\tau_2\psi^c$ and for the pions $G\boldsymbol{\varphi}G^{-1}=-\boldsymbol{\varphi}$. This last relation is responsible for the known rule which forbids an even number of pions to go into an odd number of pions if G is conserved. Moreover G satisfies the relations $[G, N]_-=0$, $[G, S]_+=0$, $G^2=(-)^{N-S}$ $[G, \tilde{I}]=0$, where N is the nucleon number, S the strangeness and $\boldsymbol{I}$ the total isotopic spin operator. We next consider the bilinear expression $(\psi\Gamma_i\tau_+\psi)$ and we find $G(\psi\Gamma_i\tau_+\psi)G^{-1}=-\xi_i(\psi\Gamma_i\tau_+\psi)$, where $\xi_i=+1$ for $i=S, A, P$ and $\xi_i=-1$ for $i=V, T$. Now we say that all covariants Q_i which transform according to $GQ_iG^{-1}=-\xi_iQ_i$ namely in the same way as the simple $(\psi\Gamma_i\tau^+\psi)$ of the same tensor character i, are of the first class; those which transform according to $GQ_iG^{-1} = \xi_i Q_i$ are of the second class. Let us now go back to the strangeness conserving current, and consider for instance its vector part. The first term $(\psi\gamma_\mu\tau_+\psi)$ is of the first class, by definition. When however, we want to add a similar term for the Σ-particles we find that we can chose among all the possible linear combinations of the form $a\Sigma^+\gamma_\mu\Sigma^0 + b\Sigma^0\gamma_\mu\Sigma^-$. If, however, we require that it behaves as a first class covariant vector, then we find that $a = -b$. This means that we can write our term for the Σ-particles as $\Sigma\gamma_\mu T_+\Sigma$, where T_+ is the appropriate isotopic spin matrix. One finds in general that one is led to the conclusion that the selection rules for a vector strangeness conserving current consisting of the pairs [5] are those of the component of a vector in isotopic spin space. This holds also for the total current including the axial which can then be constructed from the requirement that fermions appear in the projection $\frac{1}{2}(1+\gamma_5)$.

If one now assumes as a principle that in the strangeness-conserving current there are only first class quantities, how can one verify such a principle experimentally? In β-decay there is generally not much hope, because a term such as $\langle p|\bar{\psi}_p\gamma_\mu\psi_n|n\rangle$, where $\langle p|$ and $|n\rangle$ are the physical states is always well approximated by a term of the form $u_p\gamma_\mu u_n$ between free spinors, because of the low momentum transfer and similarly for $\langle p|\bar{\psi}_p\gamma_\mu\gamma_5\psi_n|n\rangle$. One has to study energetical β-decay to discover such effects. In $\mu^- + p \to n + \nu$ the momentum transfer is larger. A matrix element $\langle p|\psi_p\gamma_\mu\psi_n|n\rangle$ will be expressible in general as

$$\langle p|\bar{\psi}_p\gamma_\mu\psi_n|n\rangle = a_V(K^2)\gamma_\mu + b_V(K^2)K_\nu\sigma_{\mu\nu} + ic_V(K^2)K_\mu\,, \tag{25}$$

where K_μ is the momentum transfer $K_\mu = K_{p,\mu} - K_{n,\mu}$ and a, b, c are invariant form factors. Similarly

$$\langle p|\psi_p\gamma_\mu\gamma_5\psi_n|n\rangle = a_A(K^2)i\gamma_\mu\gamma_5 + b_A(K^2)K_\mu\gamma_5 + ic_A(K^2)K_\nu\sigma_{\nu\mu}\gamma_5\,. \tag{26}$$

One can immediately convince himself that in each of the above expressions the first two terms behave as first class quantities. In fact, as we said before, V and T take on the same ξ_i and also A and P take on the same ξ_i. The last terms in each of the two expressions are instead of the second class. In $\mu^- + p \to n + \nu$ the term $ic_V K_\mu$ contributes an effective scalar interaction of magnitude $\sim m_\mu c_V(K^2)$. A guess about c_V would suggest that it is $\sim (m_\mu/m_N)a_V$. Observation of this scalar term would indicate the presence of second class quantities. Furthermore since the nucleon contributions to the current are always of the first class (they must always be of the form $(\psi\Gamma_i\tau_+\psi)$) observation of such a scalar term would inform us about the hyperon contributions to the nuclear structure. The term $ic_A K_\nu\sigma_{\nu\mu}\gamma_5$ contributes in $\mu^- + p \to n + \nu$ an

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effective derivative coupling $ic_A(K^2)(K_\lambda^{(\mu)}+K_\lambda^{(\nu)})\sigma_{\mu\lambda}\gamma_5$. The observation of the first two terms in each of the above expressions does not tell anything about second class quantities, but it tells us of nucleon structure. The term $b_V K_\nu \sigma_{\nu\mu}$ introduces an effective derivative coupling in $\mu^- + \mathrm{p} \rightarrow \mathrm{n} + \nu$, while the term $b_A K_\mu \gamma_5$ introduces an effective pseudoscalar. GOLDBERGER and TREIMAN have used dispersion relation techniques to estimate the effective pseudoscalar and they find that it is eight times larger than the original axial part. We shall discuss later this result when speaking of the nuclear interaction of μ-particles.

In concluding this discussion about the behaviour in isotopic spin space we want again to state the result that both $\Delta I = \frac{1}{2}$ and $\Delta I = \frac{3}{2}$ contributions are expected to be present in the decay of strange particles according to this theory. There seems to be no way of avoiding this conclusion in the present formulation. If one wants an overall $\Delta I = \frac{1}{2}$ rule for the interaction, then one should introduce neutral currents into the picture. We have already seen that a strangeness conserving neutral current may certainly be introduced, but its couplings with leptons seem to give rise to some inconsistencies.

7. - Let us go back to the strangeness-conserving current. To discuss the question of the non-renormalization of the vector part we have to review first the main steps leading to the weak interaction Hamiltonian of the F.G.M.S. theory. The main assumption is that the fields always appear in the interaction with the projection $\frac{1}{2}(1+i\gamma_5)$ (we are using Feynman's notations). Let us call such a projection operator $\boldsymbol{a} = \frac{1}{2}(1+i\gamma_5)$ and let us also define the hermitian conjugate $\bar{\boldsymbol{a}} = \frac{1}{2}(1-i\gamma_5)$. One easily verifies that $\boldsymbol{a}^2 = \boldsymbol{a}$, $\boldsymbol{a}\boldsymbol{a} = 0$. The most general local non-derivative interaction Hamiltonian for β-decay will then be of the form

$$\sum_i g_i(\overline{\boldsymbol{a}\psi}_\mathrm{n}\Gamma_i\boldsymbol{a}\psi_\mathrm{p})(\overline{\boldsymbol{a}\psi}_\nu\Gamma_i\boldsymbol{a}\psi_\mathrm{e})\,. \tag{27}$$

Using $\overline{\boldsymbol{a}\psi} = \bar{\psi}\,\bar{\boldsymbol{a}}$ one can write (27) as

$$\sum_i g_i(\bar{\psi}_\mathrm{n}\bar{\boldsymbol{a}}\Gamma_i\boldsymbol{a}\psi_\mathrm{p})(\bar{\psi}_\nu\bar{\boldsymbol{a}}\Gamma_i\boldsymbol{a}\psi_\mathrm{e})\,. \tag{28}$$

Now $\bar{\boldsymbol{a}}\Gamma_i\boldsymbol{a} = 0$ for $i = S, P, T$ and $= \Gamma_i\boldsymbol{a}$ for $i = A, V$ so that (28) can be written as

$$\sum_{i=V,A} g_i(\bar{\psi}_\mathrm{n}\Gamma_i\boldsymbol{a}\psi_\mathrm{p})(\bar{\psi}_\nu\Gamma_i\boldsymbol{a}\psi_\mathrm{e})\,. \tag{29}$$

Furthermore one verifies easily that $\Gamma_V\boldsymbol{a} = \gamma_\mu\boldsymbol{a}$ is also equal to $\Gamma_A\boldsymbol{a} = i\gamma_\mu\gamma_5\boldsymbol{a}$ so that (29) can be written as

$$\sqrt{8}\,G(\bar{n}\gamma_\mu\boldsymbol{a}\boldsymbol{p})(\bar{\nu}\gamma_\mu\boldsymbol{a}\boldsymbol{e})\,. \tag{30}$$

This coupling is nothing else than than $A-V$, in the usual notations. It has been supposed here that the nucleons rather than the antinucleons have the projection $\boldsymbol{a}$ in front. Such an assignement is quite arbitrary of course and no theoretical justification can be given for it. It can however be shown to be the right one experimentally by the following argument. If antinucleons instead of nucleons have the projection $\boldsymbol{a}$ in front, the β-decay Hamiltonian would be

$$\sqrt{8}\,G(\overline{\psi}_{\bar{\mathrm{p}}}\gamma_{\mu}\overline{\boldsymbol{a}}\psi_{\bar{\mathrm{n}}})(\overline{\psi}_{\nu}\gamma_{\mu}\boldsymbol{a}\psi_{\mathrm{e}})\,. \tag{31}$$

By using the properties of the charge conjugation matrix C and the relations $\psi^{C}=C\overline{\psi}^{T}$, $\overline{\psi}^{C}=C^{-1}\psi^{T}$, one can rewrite the above Hamiltonian as

$$\sqrt{8}\,G(\overline{\psi}_{\mathrm{n}}\gamma_{\mu}\overline{\boldsymbol{a}}\psi_{\mathrm{p}})(\overline{\psi}_{\nu}\gamma\boldsymbol{a}\psi_{\mathrm{e}})\,. \tag{32}$$

This is the $A+V$ combination. Now, in the decay of polarized neutron $A-V$ would give no asymmetry (neglecting the small deviations from the equality of the axial and of the vector coupling constants which are due to renormalization effects), while $A+V$ would give maximum asymmetry (also in the same approximation). The observed asymmetry is consistent with $A-V$, thus suggesting the form (40) for the β-decay Hamiltonian and the assumption that nucleons have the factor $\boldsymbol{a}$ in front and antinucleons the factor $\overline{\boldsymbol{a}}$. To build up the current J_{μ} one adds the contributions from the pair (pn), (νe), (νμ), etc.

$$J_{\mu}=(\overline{p}\gamma_{\mu}\boldsymbol{a}n)+(\overline{\nu}\gamma_{\mu}\boldsymbol{a}\mu)+(\overline{\nu}\gamma_{\mu}\boldsymbol{a}e)\,. \tag{33}$$

It has been assumed here that μ^{-} is a particle. The other choice would have led to a μ-decay Hamiltonian given by $\sqrt{8}\,G(\overline{\nu}\gamma_{\mu}\boldsymbol{a}\overline{\mu})(\overline{\nu}\gamma_{\mu}\boldsymbol{a}e)$ which would lead to a ϱ-value of zero in contradiction to experiment. That the electron spectrum should go down to zero independent of the statistical weight at the maximum electron energy if two neutrinos or two antineutrons are emitted follows from the Pauli principle. In fact at the maximum electron energy the two neutrinos would travel in the same direction, although in general with different momenta. The wavefunction of a two component neutrino does not however contain any indication of the magnitude of its momentum except in the exponential factor which contributes to the δ-function expressing the total momentum conservation, and since the decay amplitude has to vanish when the two momenta are equal it will also vanish in the general case. So at maximum electron energy the decay amplitude has to go to zero by virtue of the Pauli principle. From the μ-meson decay Hamiltonian $\sqrt{8}\,G(\overline{\mu}\gamma_{\mu}\boldsymbol{a}\nu)(\overline{\nu}\gamma_{\mu}\boldsymbol{a}e)$ one finds for the μ-meson lifetime the expression $\tau=192\pi^{3}/G^{2}m_{\mu}^{5}$ and noting that no appreciable effects are expected here to renormalize the value of the

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coupling constants (the largest radiative corrections arise from virtual photons) one can use this expression to calculate G the bare coupling constant of the weak interaction Hamiltonian. Surprisingly one finds that the coupling constant so determined is identical, in the limits of the small experimental errors, to the effective coupling constant for the Fermi part of β-decay, that one can measure from the lifetime of ^{14}O. Namely from the lifetimes of ^{14}O one finds $G=(1.01\pm0.01)\cdot10^{-5}(1/m_N^2)=(1.41\pm0.01)\cdot10^{-49}$ erg/cm^3 and inserting such a value into the μ-lifetime formula one obtains $\tau=(2.22\pm0.02)\cdot10^{-6}$ s. This agreement seems to suggest strongly that the hypothesis of universality holds, namely that the formulation with a current of the form (33) is perhaps right, and furthermore, it indicates that, surprisingly, the renormalization effects due to strong interactions are absent for the V part of β-decay. As far as this latter circumstance is concerned, one would think that, unless some unknown accident takes place, a symmetry principle is operating at this place in a way similar to gauge invariance in quantum electrodynamics which leads to the non-renormalization of the electric charge e. Gauge invariance implies that in the zero energy limit the charge of the proton is rigorously e and furthermore it insures that the result holds independent of the structure of the nucleon. We may say that physically this happens because when a proton emits a virtual π^+ to become a neutron the virtual π^+ must, because of gauge invariance, interact with the photon field and again exhibit the charge $+e$. It is only for smaller wavelengths that are able to explore the structure, that an effective charge different from e may be exhibited. From this analogy it will not appear surprising that, in order to have non-renormalization for the V coupling in β-decay one has to introduce a direct interaction of the pion field bilinearly with the lepton pairs. We shall later discuss fully the formal problem that is involved here and the physical implications of the scheme. We would like however to show first how the argument given before for the non-renormalization of the vector part, namely from the equality of the coupling constant from ^{14}O to that from μ-decay, would be invalidated if the present evidence for a ϱ-value in μ-decay definitely different from $\frac{3}{4}$ would be confirmed. It is therefore very important at this point to have a very accurate measurement of the ϱ-value in the μ-decay.

8. – Let us write the μ-decay Hamiltonian in the form

$$\sqrt{8}\,G(\overline{\mu}\gamma_\lambda\nu_L)(\bar{\nu}\gamma_\lambda e)\,, \tag{34}$$

where $\nu_L=\boldsymbol{a}\boldsymbol{\nu}$ is the lefthanded neutrino of the F.G.M.S. theory. The ϱ-value from such an interaction is exactly $\frac{3}{4}$. To explain its deviations from $\frac{3}{4}$ suggested by the measured value $\varrho=0.68\pm0.02$ (radiative corrections included, the value is due to CROWE) we follow the Lee-Yang proposal [12] that the interaction has a more complicated structure and does not occur at a single space-time point. The above Hamiltonian will therefore be modified to

$$\sqrt{8}\,G(\bar{e}\gamma_\mu\nu_L(x))\,K(x-x')(\bar{\nu}_L\gamma_\mu\mu(x'))\,, \tag{35}$$

where a form factor $K(x-x')$ has been introduced. Of course (35) will then be generalized to the expression for the interaction of J_μ with itself, which will look something like $J_\mu(x)K(x-x')J^+_\mu(x')$. This form is not the most general one since one could still have something looking as $J_\mu[f(K^2)+K_\mu K_\nu g(K^2)]J^+_\mu$; however we expect (35) to be at least a good approximation because of the smallness of the electron mass. We do not assume here that the form factor is due to the meson X. It may originate from any complicated cause, or even reflect a true elementary non-locality. If K is expanded in a series of derivatives of the δ-function [12]

$$K(x-x') = \delta^4(x-x') + \frac{\eta}{m_\mu^2}\frac{\partial}{\partial x_\varrho}\frac{\partial}{\partial x_\varrho}\delta^4(x-x') + \dots\,, \tag{36}$$

we find for the electron spectrum from polarized muons

$$\begin{aligned}\mathrm{d}^2N \propto x^2\,\mathrm{d}x\,\mathrm{d}\xi\big\{&6-4x-\tfrac{2}{5}\,\mathrm{Re}\,(\eta)(9-16x+5x^2)+\\&+\xi(1+\tfrac{6}{5}\,\mathrm{Re}\,(\eta))/(1+\tfrac{3}{2}\,\mathrm{Re}\,(\eta))[2-4x-\tfrac{2}{5}\,\mathrm{Re}\,(\eta)][6-12x+5x^2]\big\}\,,\end{aligned}$$

where $\xi = \cos\theta$ is the angle between the muon spin and the electron momentum and x is the ratio of the momentum of the electron to its maximum energy. The energy spectrum turns out to be

$$\mathrm{d}N \propto x^2\,\mathrm{d}x\big\{3-2x-\tfrac{1}{5}\,\mathrm{Re}\,(\eta)(9-16x+5x^2)\big\}. \tag{37}$$

A ϱ-value from this spectrum is obtained by a least square fit to a Michel's spectrum. The relation is

$$\tfrac{3}{5}\,\mathrm{Re}\,(\eta) = -1.8(\tfrac{3}{4}-\varrho_{\text{exp}})\,. \tag{38}$$

One sees from (38) that $\mathrm{Re}\,(\eta)$ turns out to be negative thus excluding an intepretation of the non-locality as due to a virtual boson. The lifetime can also be calculated and it comes out to be

$$\tau^{-1} = \tau_0^{-1}(1+\tfrac{3}{5}\,\mathrm{Re}\,(\eta))\,, \tag{39}$$

and by substitution of (38) into (39)

$$\tau^{-1} = \tau_0^{-1}[1-1.8(\tfrac{3}{4}-\varrho_{\text{exp}})]\,. \tag{40}$$

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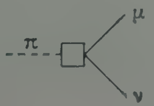
Here τ_0 is the lifetime for the local Hamiltonian ($\eta = 0$). To calculate the lifetime (40) one still needs the value of the coupling constant, contained in τ_0. This value however can be taken from the β-decay of ^{14}O if, per absurdum, we assume that the non-renormalization principle holds. The coupling constant so determined is again the same that one would obtain neglecting the non-local effects that were introduced in our universal weak interaction. This because the momentum transfer in β-decay is very low and the non-locality is not yet felt. Taking the coupling constant from ^{14}O and $\varrho = 0.68 \pm 0.02$, and computing errors, we find the predicted lifetime for μ-decay to be $\tau = (2.60 \pm 0.13)\cdot 10^{-6}$ s. The previous surprising agreement with the experimental value has now changed into a disagreement of at least 10%.

There is one point in which the above argument may be slightly incorrect. Namely the experimental ϱ-value used here has been obtained after substracting radiative corrections as calculated for point interaction. It is impossible to estimate what the radiative corrections would become in the non-local theory, unless one makes a very definite model of the non-locality (for instance intermediate boson). In general, however, one does not expect great changes from these effects and therefore we would conclude that it is certainly very important at this point to have a very accurate determination of the ϱ-value.

The equality of the effective coupling constant for β-interaction as derived from the lifetime of ^{14}O to the true coupling constant derived from the μ-meson lifetime suggests first that there may be strict universality in the primary interactions, meaning that the coupling constants may be all equal, and, second, it suggests the absence of renormalization effects and a possible symmetry explaining such absence. It appears therefore to be important to have direct separate tests of the hypothesis of universality, possibly independent of the question of the non-renormalization of the constant by the strong interactions. The most direct test for universal interaction is provided by the ratio of $\pi \to e+\nu$ to $\pi \to \mu+\nu$ [13]. The ratio can be calculated uniquely independently of assumptions on the strong interactions for any form of the weak coupling, provided it is assumed that the two leptons emerge from the same point. For S, V, T both decays are forbidden, for P_s the ratio r,

$$r = \frac{w(\pi \to e+\nu)}{w(\pi \to \mu+\nu)}, \tag{41}$$

is 5.4, for A it is $13.6\cdot 10^{-5}$. To illustrate the argument let us consider the case of A interaction. The decay is represented by the black box



The momenta of the two leptons are assumed to appear only in the combination $p_L^\mu = p_e^\mu + p_\nu^\mu$, corresponding to the assumption of locality. Moreover energy momentum conservation implies $p_\pi^\mu - p_L^\mu = 0$, so that we can choose p_π^μ as the only independent four-vector allowed by kinematics. The matrix element will therefore be of the form

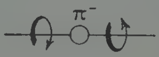
$$M \propto f(p_\pi^2)(\bar{e}\gamma_5\gamma_\mu p_\pi^\mu \nu) \propto m_e f(p_\pi^2)(\bar{\mu}\gamma_5\nu)\,, \tag{42}$$

and the ratio r_A is easily calculated

$$r_A = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = 13.6\cdot 10^{-5}\,. \tag{43}$$

The first factor $(m_e/m_\mu)^2$ is the ratio of the two squared matrix elements, the second factor is the ratio of the two phase spaces. For PS interaction the first factor has to be replaced by 1, and the ratio r_{PS} is identical to the ratio between the two space volumes.

It may perhaps be instructive to understand a little more physically the appearance of the factor m_e in the matrix element for the case relevant here of A interaction (42). Consider π^--decay. What is emitted is an l^- and a $\bar{\nu}$. The l^- is coupled in the F.G.M.S. theory as a left-handed particle, while the $\bar{\nu}$ is coupled as right-handed. Of course, it is equivalent here to talk of the F.G.M.S. theory or of a pure A interaction, as far the ratio (41) is concerned, but it is really more instructive to discuss directly in the F.G.M.S. theory. Nothing destroys the handedness of the $\bar{\nu}$ which has zero mass, and only the mass term will destroy the handedness of the l^-, since it is known that the electromagnetic interaction preserves the handedness. So in the limit of $m_l = 0$ a left-handed and a right-handed particle would have to be emitted from a zero system, which is impossible because of angular momentum conservation



Therefore the matrix element (42) has to go to zero for $m_l = 0$, as it in fact does according to (42).

What is important here is that radiative corrections (which *a priori* could not be regarded as negligible as compared to such a small decay amplitude in the case in which l⁻ is e⁻) do not change this conclusion, as long as quantum electrodynamics holds on its conventional formulation. If, however, quantum electrodynamics contained for instance a $\sigma_{\mu\nu}$ coupling, which would destroy the handedness, then the radiative corrections would not have any more to be proportional to the lepton mass, and for the case of the electron, they could

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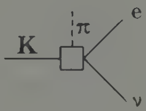
produce essential changes in the calculated ratio (43). The delicate nature of the dependence of the calculated value for r_A on the form of quantum electrodynamics was in fact pointed out by different authors [14]. The main interest in the question, at the time in which this discussion was made, came from the apparent strong disagreement between the experimental value for r and its predicted value r_A in the F.G.M.S. theory. Recent experiments have however reduced this discrepancy and are quite consistent with the theoretical value [15]. Apparently this case of the ratio of $\pi \to e+\nu$ to $\pi \to \mu+\nu$ (together with the corresponding case of $K \to e+\nu$ and $K \to \mu+\nu$) is the only case for which a test of the hypothesis of universality can be made directly without going into the various complications due to the presence of strong couplings. There is still another case, which although a little more complicated offers a similar possibility, namely the comparison of the $K \to e+\nu+\pi$ and $K \to \mu+\nu+\pi$ modes [16]. A decay mode $K \to e+\nu+\pi$ can also be represented by a black box



The structure of the black box is here a little more complicated because there is one more prong coming out from the box. One finds that in the K rest system one now needs two functions of the pion energy to describe the box. In fact 4-momentum conservation requires

$$p_K^\mu = p_\pi^\mu + p_L^\mu \,, \tag{44}$$

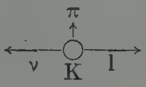
and, taking p_K^μ and p_L^μ as independent, the most general matrix element will be of the form

$$(\bar{l}\gamma_\mu P^\mu a\nu) \,, \tag{45}$$

where P^μ is a linear unknown combination of p_K^μ and p_L^μ. Using the Dirac equation one can write (47) as

$$m_e X(\bar{l} a\nu) + iY(\bar{l}\gamma_\mu p_K^\mu a\nu) \,, \tag{46}$$

where X and Y are unknown functions of $(p_K^\mu p_\pi^\mu)$, a quantity which is essentially the pion energy in the K rest system. The hypothesis of universality implies that X and Y are the same functions of the pion energy independent of l being a μ or an e. To see in a particular case how one does effectively find limitations on the decay probabilities from the hypothesis of universality, consider the particular configurations where the pion is emitted at rest



The situation is exactly the same as for π → e+ν and π → μ+ν and in fact for such configurations the ratio

$$r'_\Delta = \frac{w(\mathrm{K} \to \mathrm{e} + \nu + \pi)}{w(\mathrm{K} \to \mu + \nu + \pi)},$$

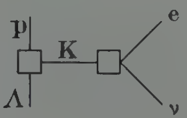
in uniquely determined:

$$r'_\Delta = \frac{m_e^2}{m_\mu^2}\left(\frac{\Delta_0^2 - m_e^2}{\Delta_0^2 - m_\mu^2}\right)^2 = 2.6\cdot 10^{-6}, \tag{47}$$

where $\Delta_0 = m_\mathrm{K} - m_\pi$. The only change with respect to the pion case has been the appropriate modification of the phase space factor. In the general case one finds the following consequences from the hypothesis of universality: if the pion energy is measured the electron energy spectrum is uniquely determined and only two possible muon spectra are allowed. For each of such two spectra the complete muon polarization (longitudinal and along the pion direction) is uniquely predicted [16]. More stringent conditions follow if it is supposed that as a first approximation the energy dependence in X and Y can be neglected.

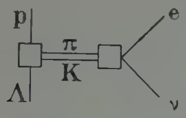
Finally, a few words about the β-decay of hyperons. If such decay modes were much more frequent than they really are, one could think of proposing tests of the universality hypothesis by comparing for instance $\Lambda^0 \to \mathrm{p} + \mathrm{e}^- + \bar\nu$ to $\Lambda^0 \to \mathrm{p} + \mu^- + \bar\nu$. The leptonic decay modes of hyperons are however surprisingly absent or very rare. If one assumes that the coupling constants are not renormalized (or at least only little renormalized as it appears in β-decay), then one expects from a direct $(\bar{\mathrm{p}}\Lambda^0)(\bar{\mathrm{e}}\nu)$ interaction, $\Lambda \to \mathrm{p} + \mathrm{e}^- + \bar\nu$ at a rate of $5.3\cdot 10^7\ \mathrm{s}^{-1}$, corresponding to 1.6% of the decays. Similarly from $(\bar\Sigma \mathrm{n})(\bar{\mathrm{e}}\nu)$, $\Sigma^- \to \mathrm{n} + \mathrm{e}^- + \nu$ would occur at a rate of $3.5\cdot 10^8\ \mathrm{s}^{-1}$, corresponding to 5.6% of the cases. No evidence has been reported so far for such decay modes. We want to make a few remarks here in this connection. First, we know that leptonic decay modes of hyperons cannot be completely absent. In fact even if there were no direct interaction of the sort $(\bar{\mathrm{p}}\Lambda^0)(\bar{\mathrm{e}}\nu)$ one still would have graphs such as

(48)



and

(48′)



and we know that the four vertices in the above graphs do exist, corresponding to the production experiments and to the K → e+ν, K → π+e+ν decays. (K → e+ν must be inferred from K → μ+ν.) Graphs of this sort will occur in a dispersion theory treatment of the process. The intermediate states are physical states. It can be shown that (48) gives a negligible pseudoscalar contribution (the K is assumed to be pseudoscalar), whereas the contribution of (48′) is quite appreciable, but its evaluation is a little uncertain. The second remark, that we want to make is that if, as we also considered, in the strangeness-non-conserving current also terms (Kπ) are present, there would be also contributions arising from $\Lambda \to p+K \to p+e^-+\nu+\pi^0 \to p+e^-+\nu$ and similar processes. However the axial contribution (no interference $A - V$ in the total rate if the electron mass is neglected!), is left unchanged. The restriction only to the $(\bar p\Lambda)(\bar e\nu)$ term is much more stringent than the hypothesis of universality. Finally, numerical coefficients may very well be present in front of the term $(\bar p\Lambda)$ in the current, depending on the way in which universality has to be formulated. Such coefficients may have to do with the question whether one really has to put the Λ-field operator into the original formulation, which has to do with universality, or whether one has instead to use particular linear combinations of the different hyperon fields. It is apparent that no answer to these questions will be possible unless the available experimental information is greatly increased. Of course one can also always invoke renormalization effects for the coupling contants to explain deviations of the data from the theoretical predictions, but it would seem to us that such justification would be a little dubious if one had to use it for all the cases in which no leptonic decay modes are observed.

9. – We can now formulate explicitly the hypothesis of non-renormalization for the vector strangeness-conserving current, according to the Feynman Gell-Mann proposal. We shall also reproduce a recent suggestion by Gell-Mann for a possible experimental test of the hypothesis [17]. Experimentally the β-decay Hamiltonian is found to be given by

$$\sqrt{8}\,G(\bar p\gamma_\mu \tfrac{1}{2}(1+\alpha\gamma_5)n)(\bar e\gamma_\mu \boldsymbol{a}\nu)\,, \tag{49}$$

where G is the same (inside experimental errors of 2%) as the μ^--decay coupling constants appearing in the interaction term

$$\sqrt{8}\,G(\bar\nu\gamma_\mu \boldsymbol{a}\mu)(\bar e\gamma_\mu \boldsymbol{a}\nu)+\text{h. c.} \tag{50}$$

The constant α is found to be $= +1.20$ [18]. The sign $+$ is determined from the experiment on the decay of polarized neutrons, which gives very small asymmetry, thus suggesting the $A - V$ form instead of the $A + V$ form. We have already discussed how the ambiguity between the $A - V$ and $A + V$ combinations could not be solved from the theory alone but had to be decided by experiment. As to the magnitude of α it can also be derived from the polarized neutron experiment, but with a rather large error. Better determinations are obtained from the $B - x$ diagram and from the free neutron lifetime. In this section we shall mainly discuss the problems arising from the result 1), namely from the equality of the renormalized vector constant in β-decay with the value of the bare coupling constants as obtained from μ-decay. The problems arising from 2), namely, why is the renormalization so small and why it is such as to increase the value of the coupling constant will be discussed in the next sections.

The Feynman Gell-Mann suggestion to explain the absence of renormalization effects in the V part of the β-decay interactions consists in postulating that the strangeness-conserving vector current is divergenceless. The absence of renormalization effects on the charge e in quantum electrodynamics is a consequence of the divergenceless of the charge current, which, for instance, for a system containing nucleons and pions is given by

$$j_\mu = j_\mu^{(N)} + j_\mu^{(3)} = \tfrac{1}{2}(\overline{\psi}\gamma_\mu\psi) + (\overline{\psi}\gamma_\mu\tau_3\psi) + i(\varphi^* T_3 \partial_\mu \varphi - (\partial_\mu\varphi)^* T_3 \varphi) \,. \tag{51}$$

Here the first term is divergenceless because it is the local nucleon number current, while the last two terms are divergenceless because they add up to the local current of the 3-component of isotopic spin. The decomposition corresponds strictly to the decomposition of Q according to

$$Q = \tfrac{1}{2}N + I_3 \tag{52}$$

for a system with zero strangeness. The interaction with the electromagnetic fields is given by

$$-ej_\mu A_\mu = -ej_\mu^{(N)} A_\mu - ej_\mu^{(3)} A_\mu \,, \tag{53}$$

and the first term contributes to the Hamiltonian a term behaving like a scalar in isotopic spin space, while the second term contributes a term behaving like the 3-component of a vector. Now one can by analogy postulate that in the vector part of the weak strangeness conserving interaction what is coupled is the $j_\mu^{(\pm)} = j_\mu^{(1)} \pm ij_\mu^{(2)}$ component of the isotopic spin current

$$j_\mu^{(i)} = \frac{1}{2}\, i\overline{N}\tau_i\gamma_\mu N + \varepsilon_{ikl}\pi_k \frac{\partial \pi_l}{\partial x_\mu} + \dots \tag{54}$$

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More explicitely the V part of the β-decay interaction is given by

$$-Gj_\mu^{(+)}l_\mu^{(+)} + \text{h. c.}, \tag{55}$$

where

$$l_\mu^{(+)} = i\sqrt{2}(\bar{e}\gamma_\mu \boldsymbol{a}\nu). \tag{56}$$

The lepton covariant $l_\mu^{(+)}$ plays here a role entirely similar to the field A_μ. The term (54) is directly comparable with the term $-ej_\mu^{(3)}A_\mu$ in (53). The matrix elements are strictly proportional. For any γ-transition due to the second term in (53) there is a corresponding transition due to (55). The Hamiltonian densities that contribute are, for the two transitions,

$$-Gj_\mu^{(+)}l_\mu^{(+)} + \text{h. c.}, \tag{57}$$

and

$$-j_\mu^{(3)}A_\mu. \tag{57$'$}$$

Let us study in detail the analogy by making a multipole expansion for both the isotopic vector part of the electromagnetic interaction and our vector weak coupling. The monopole terms are (limiting the discussion to e⁻ emission)

$$-GI_+\sqrt{2}(\bar{e}\gamma_4 \boldsymbol{a}\nu), \tag{58}$$

$$-eI_3A_0, \tag{58$'$}$$

where $\boldsymbol{I}$ is the total isotopic vector and the fields are taken at the nucleus. Equations (58) and (58′) may be used if one can neglect the energy release to the photon field. For instance one can compare

$$\pi^\pm \to \pi^0 + e^\pm + \nu, \tag{59}$$

with the β-decay mode of ^{14}O. They both are 0 → 0 transitions, so only V is involved. One has to evaluate the matrix element of I^+ between π^- and π^0, and between ^{14}O and its daughter nucleus. This can be done only by isotopic spin considerations and one actually finds that $\pi^\pm \to \pi^0 + e^\pm + \nu$ and ^{14}O have the same ft value. The phase space available in (59) is however so small that to make any experiment to determine its rate would be very difficult. In general in any $0^+ \to 0^+$ β-transition, which is allowed and only involves the interaction, the matrix element will be that of I_+ in the approximation in which (58) holds. This is however not a strong prediction of the theory. It would automatically be satisfied in any nuclear model in which the nucleus is described as a system of dressed nucleons. For a possible verification of the theory Gell-Mann suggests accurate measurements of some high energy β-transitions. For instance to the γ-decay of the 15.11 MeV excited level of ^{12}C (with $J=1^+$, $I=1$) to the ground state of ^{12}C $J=0^+$, $I=0$) there corresponds, in the sense of the above analogy, the β-decay of the two other members of the $I=1$ supermultiplet (^{12}B and ^{12}N) to the ground state of ^{12}C. The V parts for such transitions can therefore be expressed in terms of the γ-decay rate. The main contribution is of course due to the A part, but the V contribution can still be estimated to be $\sim 10\%$ if the non-renormalization hypothesis holds. We note that the non renormalization hypothesis may also be useful in other connections. For instance it will give a relation between the V contribution of $\pi \to \mu+\nu+\gamma$ and $\pi^0 \to \gamma+\gamma$, and the V contribution of $\Sigma^+ \to \Lambda^0+e^++\nu$ to $\Sigma^0 \to \Lambda^0+\gamma$. It is useful to note that there is no interference in the total rate between A and V in $\Sigma^+ \to \Lambda^0+e^++\nu$ as long as the electron mass is neglected.

10. – We shall now discuss the problem of the renormalization of the axial vector coupling constant. As we said before the Gamow-Teller coupling constant in β-decay is by now definitely different from the Fermi coupling constant and thus one should not insist further in trying to make the axial current also non-renormalizable. Moreover it might be difficult, mathematically, to make the axial current divergenceless. The two questions may however be quite unrelated. One has an argument due to GOLDBERGER and TREIMAN which shows that a divergenceless axial current would be in conflict with established experimental evidence in β-decay [19]. Let us sketch here the argument. Write the axial β-decay matrix element in the form

$$(\bar{u}_e \gamma_\mu \boldsymbol{a} u_\nu)\langle p|j_\mu^{(A)}|n\rangle\,. \tag{60}$$

Now $\langle p|j_\mu^{(A)}|n\rangle$ can be written as

$$\langle p|j_\mu^{(A)}|n\rangle = (\bar{u}_p[a(K^2)\gamma_\mu\gamma_5 + ib(K^2)K_\mu\gamma_5]u_n)\,, \tag{61}$$

where $K_\mu = K_\mu^{(p)} - K_\mu^{(n)}$ is the momentum transfer, $a(K^2)$ and $b(K^2)$ are nucleon form factors. The form (61) follows from Lorentz invariance, time reversal, and charge independence. In particular a possible term of the form $\bar{u}_p\sigma_{\mu\nu}\gamma_5K_\nu u_n$ does not appear because of the circumstance that n and p belong to the same charge multiplet. Using the Dirac equation for the leptons and momentum conservation, one finds for the A-matrix element of β-decay

$$a(K^2)(\bar{u}_e\gamma_\mu \boldsymbol{a} u_\nu)(\bar{u}_p\gamma_\mu\gamma_5u_n) + m_e b(K^2)(\bar{u}_e \boldsymbol{a} u_\nu)(\bar{u}_p\gamma_5u_n)\,. \tag{62}$$

The effective A coupling constant is $g_A = a(0)$, because of the smallness of

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the momentum transfer, and moreover there is an effective PS coupling constant $g_{\text{p}} = m_{\text{e}} b(0)$. If now we postulate

$$\frac{\partial j_{\mu}^{(A)}}{\partial x_{\mu}} = 0\,, \tag{63}$$

we find in momentum space

$$K_{\mu} \langle p | j_{\mu}^{(A)} | n \rangle = 0\,. \tag{64}$$

Substituting (61) in (64) we find $(u_{\text{p}}[a(K^2)\gamma_{\mu} K_{\mu} \gamma_5 + ib(K^2)K^2\gamma_5]u_{\nu}) = 0$ and using the Dirac equation for the nucleons, we finally find the relation

$$g_{\text{p}} = -\frac{2m_N m_{\text{e}}}{K^2}\, g_A\,, \tag{65}$$

between the PS and the A coupling constants. The ratio $g_{0\text{p}} g_A$: turns out then to be energy dependent, because of the K^2 dependence, and very large, $\sim 1\,000$ at least. This is inconsistent with experiment. No such difficulties arise from the V part. Here the matrix element analogous to the (61) contains only a γ_{μ} term and a term $\sigma_{\mu\nu} K_{\nu}$. The term with K_{ν} alone disappears by invariance arguments. It is then evident that the condition (64) can indeed be satisfied. There is perhaps a more direct argument to show that the axial strangeness conserving current is not divergenceless. In fact the decay modes $\pi \to \mu + \nu$, $\pi \to e + \nu$, would not occur through such a divergenceless current. Since they both exist, the A current cannot be divergenceless.

Finally let us speculate about the quantitative aspect of the renormalization problem for the axial coupling constant. If one accepts the Feynman-Gell-Mann suggestion that the vector part is non-renormalizable, one is then left with the conclusion that the renormalized axial coupling constant is α times larger than the bare coupling constant (with $\alpha \simeq 1.20$). The first question concerns the sign of the renormalization effect. Why is the effective coupling constant larger than the bare coupling constant? We first should point out that the situation here has not very much to do with single cases where essentially only one field is involved. One likes to imagine that an original bare charge polarizes the vacuum and is then surrounded by charges of the opposite sign, so that the total effective charge is diminished. Here, however, the situation is certainly quite different. More appropriately one can discuss the effect in the static source theory. One can easily convince himself that the renormalization factor which multiplies the bare coupling constant to give the renormalized coupling constant is something like $(1 + \frac{1}{3}X^2)(1 + 3X^2)$, where X^2 is the probability of finding in the cloud a pion of given charge. This factor is certainly less than unity. So one is led to the conclusion that, if the renormalization effects are such as to increase the value of the coupling constant, a more sophisticated description of the nucleon is necessary to evaluate them, in particular nucleon-antinucleon pairs will play presumably an important role.

For a discussion of the problem of the renormalization of the axial coupling constant the best basis is offered by the dispersion relation approach, as developed in the particular case by GOLDBERGER and TREIMAN in their work on form factors for weak interactions [19]. We shall here only limit ourselves to a few general remarks. First of all we should point out that the problem of the renormalization of the coupling constant is perhaps undefined in the ordinary renormalization theory. Actually a renormalization program can be carried through consistently for strong electromagnetic and weak interactions, as long as the last ones are only taken up to their first order, as discussed in detail by WEINBERG [20]. To be definite let us limit the discussion to a system consisting of nucleons, pions, photons, electrons and neutrinos. There is a bare AV coupling between n, p, e and ν, which has to be taken only at first order. To have a renormalizable theory one finds that apart from the renormalized A and V coupling constants one still has to introduce two other coupling constants corresponding to two possible normalization counterterms, so that a total of four weak constants in needed. The two counterterms contribute direct amplitudes to

$$\pi \to \pi + e + \nu \tag{66}$$

and to

$$\pi \to e + \nu\,. \tag{67}$$

If one introduces a non-renormalization hypothesis for the V part one finds that one needs only a constant for (67), because the coupling constant for the counterterm producing (66) is now fixed by the condition of zero divergence of the V current. In any case however the amplitude for (67) is unrelated to the β-decay amplitude. Now in the dispersion relation approach, as developed by GOLDBERGER and TREIMAN, only two *weak* constants are instead introduced: one is the renormalized axial coupling constant, the other one is the renormalized vector coupling constant. They correspond to two subtractions, one made in the dispersion relation for the nucleon axial form factor, the other one made in the dispersion relation for the nucleon vector coupling constant. The amplitude for (67) instead is assumed to satisfy a non-subtracted dispersion relation. Thus, one can say at least that the dispersion relation techniques make it possible to give unambiguous answers to questions which could not be consistently formulated in perturbation theory unless by recourse to arbitrary cut-off procedures. An example is provided by the recent formulation by CINI, FERRARI and GATTO [21] of the problem of the neutron-proton mass difference. In renormalization theory the physical masses are directly introduced through suitable counterterms. With the dispersion relation techniques the problem of calculating the mass difference can instead be formulated unambiguously, provided the convergence of the dispersion relations is assumed. Now to formulate the problem of the axial vector renormalization one can use the dispersion relation approach, but one has to postulate that non-subtracted dispersion relations are valid for the axial and for the vector nucleon form factors for β-decay. Such a postulate is of course highly questionable, and we shall not go into more details here on these problems. Furthermore to evaluate the integrals on the absorbtive parts in a non-subtracted dispersion relation may be a very difficult question, because of the poorer convergence of the integrals, even if they converge at all, which may require a knowledge of the imaginary parts up to very high energies.

11. – Let us now summarize what we have seen about the strangeness conserving current $\mathcal{S}^{\mu}_{0}$. The vector part may very well be non-renormalizable. There is a difficult with the ϱ-value, but here one has to wait for better experiments. The axial part has almost no chance of being non-renormalizable because of the difference in the Fermi and Gamow-Teller β-decay coupling constants. Also, the A current is not divergenceless because of the existence of $\pi \to \mu + \nu$ and $\pi \to e + \nu$, and because of the large unobserved pseudoscalar in β-decay that one should have in such a case. Now what about the strangeness non-conserving current? If we say that the K is pseudoscalar then the A part of the strangeness non-conserving current $\mathcal{S}^{\mu}_{-1}$ is certainly not free of divergences because of the existence of $K \to \mu + \nu$, which occurs through the A part and implies a divergence. For the V part one can hope to have experimental information from the $K \to l + \nu + \pi$ spectra. Such decay modes infact entirely through V and the condition on the current to be divergenceless imposes restrictions on the form factors for such decay. This has been discussed by GOLDBERGER and TREIMAN [19] and also by WEINBERG *et al.* [22]. The last authors however do not use explicitely the hypothesis of universality and thus they obtain less stringent requirements. The matrix element for $K \to l + \nu + \pi$, equation (4), is written in the form

$$(l\gamma_\mu a\nu)\langle\pi|V_\mu|K\rangle\,, \tag{68}$$

where $\langle\pi|V_\mu|K\rangle = c(K^2)p_\mu + d(K^2)K_\mu$, if $p_\mu = p_\mu^{(K)} + p_\mu^{(\pi)}$. and $K_\mu = p_\mu^{(K)} - p_\mu^{(\pi)}$ The hypothesis of absence of divergences would imply $K_\mu\langle\pi|V_\mu|K\rangle = 0$, thus giving a relation between d and c, namely $K^2 d = (m_K^2 - m_\pi^2)c$. Therefore only one form factor is unknown. It can be shown that the decay rate for $K \to e + \nu + \pi$ has then necessarily to be larger than that of $K \to \mu + \nu + \pi$.

This conclusion is not inconsistent with experiment. More stringent tests along the lines of reference [16] are still possible however, but one will have to wait for better data. We should like, however, to point out that it may in fact be difficult to formulate theoretically a conservation law for the vector part of the strangeness non-conserving current. For the strangeness conserving current we employed conservation of isotopic spin. However the strangeness non-conserving current will behave in general like a superposition of $I=\frac{1}{2}$ and $I=\frac{3}{2}$, and no similar possibility can be thought of in this case. A final remark concerns the possibility of including in the scheme for the strangeness conserving terms beside the I_+ and I_- currents of isotopic spin, also the neutral I_3 current, which could be coupled to the neutral lepton class $\mathcal{L}^{(0)}$ according to the scheme we have already proposed. The only motivation for such a scheme so far would be its greater symmetry. Furthermore one would perhaps have some new possibility open for the relevant question of defining in a local way rototations in isotopic spin space, a question that was explored some time ago by Yang and Mills [23].

12. – It is apparent from our discussion that many points still remain to be clarified and more accurate experiments are needed in many cases. The formulation of the weak interactions in terms of the interaction of a current with itself may eventually turn out to be correct. However the explicit definition of the current is still unknown or at least partialy known. The vector strangeness-conserving current, if it has to be divergenceless, as suggested by the absence of renormalization effects, is determined uniquely as the plus component of the isotopic spin current. A minimal axial strangeness-conserving current is then determined from the condition that fermions always appear in the projection $\frac{1}{2}(1+\gamma_5)$. As far as the strangeness non-conserving current is concerned, no such general symmetries have been discovered so far, to provide us with an unambiguous definition. In particular boson terms of the form $(K\pi)$ cannot be excluded. It must also be remarked that the structure of such current is simplified considerably if a $\Delta S=\pm 1$ selection rule holds for the interaction, or even more restrictive conditions hold such as a $\Delta T=\frac{1}{2}$ rule for the current. We have also discussed the possibility of neutral currents. A very symmetric scheme would be obtained by introducing a neutral strangeness conserving current to be identified essentially with the 3 component of the isotopic spin current, possibly coupled also to the leptons, with the exclusion of the (μe) pair. Such a scheme would lead to no experimental inconsistencies so far, and could perhaps provide new possibilities to define isotopic spin rotations in a local way. If a $\Delta I=\frac{1}{2}$ rule holds in the decay of strange particles the scheme of universal interaction as the coupling of a charged current to itself will have to be revised. A coupling of a neutral strangeness non-conserving current to leptons seems however inconsistent with

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present data. We should furthermore point out the necessity of accurate experimental tests of the non renormalization hypothesis for the vector part. More theoretical and experimental accuracy is required in the comparison of the μ-decay constant with the effective vector coupling constant of β-decay. The necessity of an accurate determination of the ϱ-value in μ-decay is connected with this problem, and also with the question of the possible existence of an intermediate meson to act as an intermediate agent of weak interactions. A more accurate upper limit for $\mu \to e+\gamma$ would also be useful in this connection. The apparent absence of hyperon β-decays may be relevant for the definition of the strangeness non-conserving current. However the Λ and Σ lifetimes and the $K \to \mu+\nu$ and $K \to \mu+\nu+\pi$ rates are all well understood in terms of coupling constants of the order of that for $\pi \to \mu+\nu$. A very difficult theoretical problem is presented by the sign and smallness of the renormalization of the β-decay axial coupling. To compute such effects one would have to rely on theoretically doubtful non-subtracted dispersion relations, and a possible evaluation would moreover be very difficult because of the many unknown amplitudes which may contribute. It is important to note that a theory renormalizable up to the first order in the weak interactions would require the presence of counterterms which are not strictly necessary if the convergence of certain subtracted dispersion relations can be assumed. More simple discussions on the role of the strong interactions in the weak processes can be carried out in the comparisons of processes in which the same structures appear for different values of the momentum transfers, for instance in the comparison of μ absorption to β-decay, or of $K \to \pi + e + \nu$ to $K \to \pi + \mu + \nu$. For such comparison one can use substracted dispersion relations, whose consequences are perhaps more reliable.

REFERENCES

[1] R. P. Feynman and M. Gell-Mann: *Phys. Rev.*, **109**, 193 (1958); E. C. G. Sudarshan and R. E. Marshak: *Proc. of the Padua-Venice Conf.* (Sept. 1957).

[2] M. Gell-Mann: *Suppl. Nuovo Cimento*, **4**, 847 (1956).

[3] This form of the amplitude follows from time reversal invariance (R. Gatto: *Phys. Rev.*, **106**, 168 (1957)). Independently of time reversal it can be shown to hold as a very good approximation (T. D. Lee, R. Oehme and C. N. Yang: *Phys. Rev.*, **106**, 340 (1957)).

[4] L. E. Okun and B. Pontecorvo: *Žur. Èksper. Teor. Fiz.*, **32**, 1578 (1957).

[5] A. Pais and O. Piccioni: *Phys. Rev.*, **100**, 1487 (1955).

[6] T. D. Lee and C. N. Yang: *Phys. Rev.*, **108**, 1611 (1957).

[7] R. P. Feynman and M. Gell-Mann: private communication.

[8] G. Feinberg: *Phys. Rev.*, **110**, 1482 (1958).

[9] M. Gell-Mann and A. Pais: *Proc. of the Glasgow Conf.* (London, 1955), p. 342; R. Gatto: *Nuovo Cimento*, **3**, 318 (1956); G. Wentzel: *Phys. Rev.*, **101**, 1215 (1956).

[10] R. Gatto and R. D. Tripp: *Nuovo Cimento*, **6**, 367 (1957).

[11] S. Weinberg: *Phys. Rev.* (to be published).

[12] T. D. Lee and C. N. Yang: ref. [6]; S. Bludman and A. Klein: *Phys. Rev.*, **109**, 550 (1958).

[13] M. Ruderman and R. Finkelstein: *Phys. Rev.*, **76**, 1458 (1949); S. B. Treiman and H. W. Wyld jr.: *Phys. Rev.*, **101**, 1552 (1956).

[14] R. Gatto and M. Ruderman: *Nuovo Cimento*, **8**, 775 (1958); **9**, 556 (1958); R. P. Feynman: *Geneva Conference on High Energy Nuclear Physics* (1958); J. C. Taylor: to be published.

[15] Reported by G. Bernardini, at the Geneva Conference for the peaceful Uses of Atomic Energy (1958).

[16] R. Gatto: *Phys. Rev.* **111**, 1426, (1958).

[17] M. Gell-Mann: *Phys. Rev.*, **111**, 362. (1958).

[18] Reported from L. Mikallyan at the Geneva Conference, 1958 (from the lifetime of the free neutron). The values that one obtains from a diagram are more consistent with a smaller value (1.15).

[19] M. Goldberger and S. B. Treiman: *Phys. Rev.*, **111**, 354 (1958).

[20] S. Weinberg: *Phys. Rev.*, **106**, 1301 (1958).

[21] M. Cini, E. Ferrari and R. Gatto: *Phys. Rev. Letters*, **2**, 7, (1959).

[22] Reported by R. Marshak at the *Geneva Conference on High Energy Nuclear Physics* (1958).

[23] C. N. Yang and R. L. Mills: *Phys. Rev.*, **96**, 191 (1954).

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An Introduction to Dispersion Relations.

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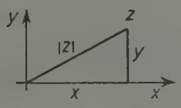
I have been carefully briefed as to the subject matter of my talks. My assignment is to impart to you the elements of dispersion relations, building up both the physical and the mathematical backgrounds of this subject, which, during the last few years, has come into prominence. On the other hand, I am not prepared to pronounce judgment on the real use of dispersion relations in meson theory, a question which will be dealt with by others during this course.

Dispersion relations can be established in a variety of problems, ranging from the theory of electrical networks [1, 2] and the optics of dispersive media [3-6] to potential scattering in non-relativistic wave mechanics [7, 8] and finally to quantum electrodynamics [9] and pion physics.

The mathematics of dispersion relations is the same in all these problems. It is essentially the theory of functions of a complex variable, and especially the famous Cauchy theorem of complex integration. This is very fortunate! On the contrary, the physical foundations of dispersion relations are certain causality conditions, which must be suitably formulated in every single problem. For this reason, I prefer to start from the mathematical angle.

The idea is that if we have a function $f(x)$ of a *real* variable x, say $f(x)=1/(x^2+m^2)$, it is often possible to extend its domain of definition, in the sense that we can find a function $f(z)$ of the complex variable $x+iy$, which, on the real axis, reduces to our given $f(x)$: $f(x)=[f(z)]_{y=0}$. In the case of $f(x)=1/(x^2+m^2)$, the function sought is clearly $f(z)=1/(z^2+m^2)$.



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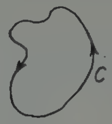
Among the functions of a complex variable, the *so-called analytic functions* play a special rôle. They can be defined in several ways, from among which we shall choose the well-known integral definition, that the integral of an analytic function along any closed curve in the domain of analyticity is zero,

$$\int_c f(z)\,\mathrm{d}z = 0\,. \tag{1}$$

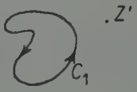


There are functions which are analytic in a certain region except at a finite number of points, where they are « singular ». *E.g.*

$$\varphi(z) = \frac{1}{z - z'}$$

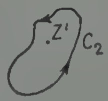
is singular at $z = z' = x' + iy'$ (simple pole). If we integrate this function along a closed curve which does *not* enclose z', things are as they were before, and the result is zero:

$$\int_{c_1} \varphi(z)\,\mathrm{d}z = 0\,. \tag{2}$$



On the other hand,

$$\int_{c_2} \varphi(z)\,\mathrm{d}z = 2\pi i\,, \tag{2'}$$



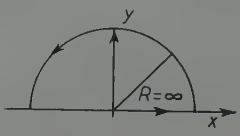
if z' is a point inside c_2 (we would have $-2\pi i$ in the right member, if c_2 were described in the clockwise direction).

If $f(z)$ is an analytic function, the product $f(z)/(z - z')$ is no longer an analytic function, unless $f(z') = 0$ (which we exclude). In this case, instead of (1) we have

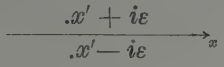
$$\frac{1}{2\pi i}\int_c \frac{f(z)\,\mathrm{d}z}{z - z'} = \begin{cases} f(z') & \text{if } z' \text{ is inside } c, \\ 0 & \text{if } z' \text{ is outside } c. \end{cases} \tag{3}$$

This is the well-known *Cauchy formula* and the whole basis of dispersion relations. These are derived as follows.

Assume that $f(z)$ is analytic in the whole upper half of the complex plane, and take as the contour of integration a curve c consisting of the real axis and of a semicircle of infinite radius in the upper half-plane:



Let z' be very near the real axis, say $z' = z^{(\pm)} = x' \pm i\varepsilon$ $(\varepsilon > 0)$



Then

$$\frac{1}{2\pi i}\int \frac{f(z)\,\mathrm{d}z}{z - z^{(\pm)}} = \begin{cases} f(x' + i\varepsilon) & (+\text{ sign}), \\ 0 & (-\text{ sign}), \end{cases} \tag{4}$$

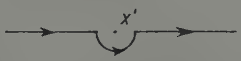
since in one case $x' + i\varepsilon$ lies inside the contour of integration, whereas in the other case it lies outside.

A dramatic simplification of this formula takes place if the modulus of $f(z)$, $|f(z)|$ tends to zero at large distances from the origin, for then the integral along the semi-circle tends to zero and we are left with an integral along the real axis

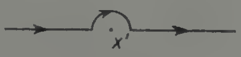
$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(x)\,\mathrm{d}x}{x - z^{(\pm)}} = \begin{cases} f(x' + i\varepsilon) & (+\text{ sign}), \\ 0 & (-\text{ sign}), \end{cases} \tag{5}$$

This is a dispersion relation, thought not yet in a satisfactory form. In fact, in the integral the function $f(x)$ occurs only for real values of its argument, and this is a good job, since the argument, in the application, will be a real physical quantity, *e.g.* a frequency or a momentum. On the other hand, the occurrence of $f(x' + i\varepsilon)$ in the right member makes it clear that we must perform the limit $\varepsilon \to 0$.

We cannot just pur $\varepsilon = 0$ in (5). For it was essential to the derivation of (5) that the point $x' + i\varepsilon$ had to be inside the contour of integration, and $x' - i\varepsilon$ outside. Moreover, for $\varepsilon = 0$ the integral in the left member becomes $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x/(x - x')$, which, unless suitably defined, is divergent at $x = x'$ by virtue of the factor $1/(x - x')$. The correct way of performing the limit $\varepsilon \to 0$ is the following. If the point $x' + i\varepsilon$, say, is to remain inside the contour of integration while $\varepsilon \to 0$, it must produce a bulge on the real axis, thus:



Therefore the path of integration along the real axis must be slightly deformed in the vicinity of x', so as to by-pass it from below. The same argument applies to $x' - i\varepsilon$, which will remain outside the contour at the expense of deforming the latter, thus:



so that x' is now by-passed from above. The indentations can be as small as we want, and are conveniently taken as semi-circles of radius ϱ ($\varrho \to 0$ in the end) and centre x'.

Therefore

$$(6) \qquad \lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty}\frac{f(x)}{x-(x'\pm i\varepsilon)}\,\mathrm{d}x = \lim_{\varrho\to 0}\left(\int_{-\infty}^{x'-\varrho}+\int_{x'+\varrho}^{+\infty}\right)\frac{f(x)}{x-x'}\,\mathrm{d}x\, +$$

$$+ \text{ non vanishing contribution from indentation} =$$

$$= P\int_{-\infty}^{+\infty}\frac{f(x)\,\mathrm{d}x}{x-x'} + \begin{cases} f(x')/2 & (+\text{ sign}),\\ -f(x')/2 & (-\text{ sign}).\end{cases}$$

Here P denotes the *principal value* of the integral. It stands for Cauchy's prescription for defining integrals of the type

$$\int\frac{f(x)}{x-x'}\,\mathrm{d}x\,,$$

by excluding from the range of integration an infinitely small interval of which x' is the middle point (this is essential!).

Using (6) we find

$$(7) \qquad f(x') = \frac{1}{\pi i}\,P\int_{-\infty}^{+\infty}\frac{f(x)}{x-x'}\,\mathrm{d}x\,.$$

NOTE 1. - The derivation of (7) from (4) can be performed symbolically by using the formula:

$$(8) \qquad \lim_{\varepsilon\to 0}\frac{1}{z\pm i\varepsilon} = P\frac{1}{z}\mp i\pi\,\delta(z)\,.$$

Then:

$$\lim_{\varepsilon\to 0}\frac{1}{x-z^{(\pm)}} = \lim_{\varepsilon\to 0}\frac{1}{(x-x')\mp i\varepsilon} = P\,\frac{1}{x-x'}\pm i\pi\,\delta(x-x')\,,$$

which, inserted in (5), yields

$$(9) \qquad \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{f(x)\,\mathrm{d}x}{x-z^{(\pm)}} = \frac{1}{2\pi i}\,P\int_{-\infty}^{+\infty}\frac{f(x)\,\mathrm{d}x}{x-x'}\, +$$

$$\pm\frac{i\pi}{2\pi i}\int_{-\infty}^{+\infty}\delta(x-x')\,f(x)\,\mathrm{d}x = \begin{cases} f(x') & (+\text{ sign}),\\ 0 & (-\text{ sign}),\end{cases}$$

and eq. (7).

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NOTE 2. - If $f(z)$ is analytic in the upper half-plane, but $|f(z)| \leqslant A|z|$ as $|z|^n \to \infty$, the function

$$F(z) = \frac{f(z) - f(x_0) - f'(x_0)(z - x_0) - \ldots - (1/n!) f^{(n)}(x_0)(z - x_0)^n}{(z - x_0)^{n+1}},$$

(x_0 real) obeys dispersion relations:

$$F(x') = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{F(x)}{x - x'} \, \mathrm{d}x ,$$

i.e.

$$f(x') = (x' - x_0)^{n+1} \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{F(x)}{x - x'} \, \mathrm{d}x + f(x_0) + \ldots + \frac{1}{n!} f^{(n)}(x_0)(x' - x_0)^n .$$

This formula may be obtained by replacing $f(z)$ in eq. (4) by $f(z)/(z - x_0 + i\eta)^{n+1}$ and using the symbolic formula [12]

$$\lim_{\eta \to 0} \frac{1}{(z - x_0 + i\eta)^{n+1}} = P \frac{1}{(z - x_0)^{n+1}} - i\pi \frac{(-1)^n}{n!} \delta^{(n)}(z - x_0) ,$$

($\delta^{(n)} = n$-th derivative of δ-function). The meaning of « P » here is that the first $(n+1)$ terms of the Taylor expansion around x_0 must be subtracted from $f(x)$ in the integral along the real axis. This point was clarified in a discussion with Drs. FERRARI and STORA.

It is customary to represent $f(x)$ as the sum

$$f(x) = D(x) + i A(x) \tag{10}$$

of its real part $D(x) = \operatorname{Re} f(x)$ and its imaginary part $A(x) = \operatorname{Im} f(x)$. Inserting (10) in (7), and comparing real terms and imaginary terms, we find

$$\left\{ \begin{aligned} D(x') &= \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{A(x)}{x - x'} \, \mathrm{d}x , \\ A(x') &= \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{D(x)}{x' - x} \, \mathrm{d}x . \end{aligned} \right. \tag{11}$$

These relations, we repeat once more, are valid if $f(z)$ is analytic in the upper half-plane and tends to zero at large distances from the origin, so that

$$\int_{\text{semi-circle of radius } R} \frac{f(z)}{z - z'} \, \mathrm{d}z \to 0 . \qquad \text{as } R \to \infty .$$

(The length of the semicircle is πR, the factor $1/(z-z')$ will contribute a factor $1/R$ as $|z|=R\gg|z'|$ along the semi-circle.)

We already have all the equipment that is necessary for establishing dispersion relations in optics. I need only remind you that for electromagnetic waves propagating in a dispersive medium, conceived as an assembly of oscillators (say N oscillators per unit volume, all with the same natural frequency ω_0 and natural width Γ) polarizability γ is

$$\gamma = N\frac{e^2}{m}\,\frac{1}{\omega_0^2-\omega^2-i\omega\Gamma}, \tag{12}$$

which is a complex function of the frequency ω. The imaginary part, as is well-known, embodies damping effects. May I remind you also of the relation $\varepsilon=1+\gamma$ between γ and the dielectric constant ε $(D=\varepsilon E=E+P=E+\gamma E)$ and $n=\sqrt{\varepsilon}\sim 1+\gamma/2$ between the refractive index and the polarizability.

Assuming this latter relation to hold exactly, we have

$$n(\omega)-1=\frac{\gamma}{2}=N\frac{e^2}{2m}\,\frac{1}{(\omega_0^2-\omega^2)-i\omega\Gamma}, \tag{13}$$

a complex function of ω,

$$n(\omega)-1=D(\omega)+iA(\omega)\,, \tag{14}$$

whose real and imaginary parts are given by

$$\begin{cases} D(\omega)=N\dfrac{e^2}{2m}\,\dfrac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\omega^2\Gamma^2}, \\ A(\omega)=N\dfrac{e^2}{2m}\,\dfrac{\omega\Gamma}{(\omega_0^2-\omega^2)^2+\omega^2\Gamma^2}. \end{cases} \tag{15}$$

So far ω has been regarded as a real quantity. However, (13) represents a function in the complex plane $\omega=\omega_1+i\omega_2$, to which all our previous considerations may be applied. In fact, $|n(\omega)-1|$ tends to zero as $|\omega|\to\infty$. Moreover, $n(\omega)-1$ has no singularities in the upper half-plane if the width Γ is taken to be independent of ω. It has two poles at the roots of the quadratic equation

$$\omega^2+i\omega\Gamma-\omega_0^2=0\,,$$

which are both in the lower half-plane. (*Note*: If the correct dependence of Γ on ω is considered, $\Gamma=e^2\omega^2/6\pi mc^3$, then the equation is cubic, and one

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of the roots is in the upper half-plane (TOLL). This is connected with the fact that causality doesn't hold for the classical electron.)

Therefore, renaming the variables, the first of eqs. (11) becomes

$$D(\omega')=\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{A(\omega)\,d\omega}{\omega-\omega'}\,, \tag{16}$$

with $D(\omega)=\mathrm{Re}\,[n(\omega)-1]$ and $A(\omega)=\mathrm{Im}\,[n(\omega)-1]=\mathrm{Im}\,n(\omega)$.

It is apparent from (15) that $A(\omega)$ is an odd function of ω, $A(\omega)=-A(-\omega)$ so that the integral can be transformed into one over positive frequencies only, and the dispersion relation for the refractive index becomes

$$\mathrm{Re}\,[n(\omega')-1]=\frac{2}{\pi}P\int_{0}^{+\infty}\frac{\omega\,\mathrm{Im}\,n(\omega)\,d\omega}{\omega^2-\omega'^2}\,. \tag{17}$$

Occasionally this equation is written in another form, by subtracting from it its value for $\omega'=0$,

$$\mathrm{Re}\,[n(\omega')-n(0)]=\frac{2}{\pi}P\int_{0}^{\infty}\frac{\omega'^2\,\mathrm{Im}\,n(\omega)\,d\omega}{\omega(\omega^2-\omega'^2)}\,. \tag{18}$$

It is now time to establish the connection between dispersion relations and causality. This can be presented in several ways. I shall proceed by enunciating a theorem which is a special case of one more general due to TITCHMARSH [10].

Let $f(x)$ be the Fourier transform

$$f(x)=\int_{-\infty}^{\infty}g(t)\exp\,[ixt]\,dt\,, \tag{19}$$

of a function $g(t)$, which is

α) bounded and square integrable,

and

β) $$g(t)=0 \qquad \text{for } t<0.$$

Then

$$f(z)=\int_{-\infty}^{\infty}g(t)\exp\,[izt]\,dt=\int_{0}^{\infty}g(t)\exp\,[izt]\,dt\,, \tag{20}$$

defines a function of the complex variable $z = x + iy$ (reducing to $f(x)$ on the real axis) which is analytic in the upper half-plane and whose real and imaginary parts satisfy dispersion relations.

Without going into mathematical details, I shall confine myself to stressing the rôle of the exponential $\exp[izt]$ which occurs in (20). Since:

$$\exp[izt] = \exp[i(x + iy)t] = \exp[ixt]\exp[-yt],$$

with $t > 0$, as the integral extends only to positive values of t, this exponential is the product of an oscillating factor $\exp[ixt]$ by $\exp[-yt]$, which latter decreases exponentially as $y \to +\infty$ (*i.e.* at large distances from the origin in the upper half-plane) and increases as $y \to -\infty$ (*i.e.* in the lower half plane). This, together with the boundedness of $g(t)$ leads to the boundedness of $|f(z)|$ in the upper half-plane, for

$$|f(z)| = \left|\int_0^\infty g(t)\exp[izt]\,dt\right| \leqslant \int_0^\infty g(t)|\,|\exp[-yt]\,dt \leqslant K\int_0^\infty \exp[-yt]\,dt =$$

$$= \frac{K}{y}\left(= \frac{K}{R\sin\theta}\ \text{along a circle}\ z = R\exp[i\theta]\right).$$

In other words, $|f(x)|$ is not only bounded in the whole upper half-plane, but indeed decreases at least as rapidly as $1/R$. Moreover, the analyticity of $f(z)$ in the upper half-plane follows from its boundedness and its being a superposition of analityc functions, *i.e.* the exponentials $\exp[izt]$. The connection of all this with Physics is clearly illustrated by the example *of absorptive condensers*. The current $I(t)$ passing through one of these is related to the applied voltage $U(t)$ by the formula:

$$I(t) = G_0 U + C_0\frac{dU}{dt} + \int_{-\infty}^{\infty} g(t-\tau)\frac{dU(\tau)}{d\tau}\,d\tau, \tag{21}$$

where $g(t)$ is a bounded function which is zero for negative values of its argument. Thus only voltage changes at times previous to t contribute to the integral, this is what is meant by « causality » in this particular problem.

For an applied voltage of the type $U(t) = U(\omega)\exp[-i\omega t]$ it is easy to show that the current is $I(t) = I(\omega)\exp[-i\omega t]$ with

$$I(\omega) = [G_0 - i\omega C_0 + i\omega F(\omega)]U(\omega), \tag{22}$$

where

$$F(\omega) = D(\omega) + iA(\omega) = \int_{-\infty}^{\infty} g(t)\exp[i\omega t]\,dt. \qquad (g(t) = 0\ \text{for}\ t < 0). \tag{23}$$

Therefore, according to Titchmarsh's theorem, dispersion relations will connect the real and the imaginary parts of the function $F(\omega)$.

The connection of causality with the analyticity properties is just as easily exhibited in the theory of optical dispersion.

Consider a signal

$$f(z,t)=\int_{-\infty}^{\infty} f(\omega)\exp\left[-i\omega\left(t-\frac{n(\omega)z}{c}\right)d\omega\right]= \tag{24}$$

$$=\int_{-\infty}^{\infty} f(\omega)\exp\left[-i\omega\left(t-\frac{z}{c}\right)\right]\exp\left[i\omega[n(\omega)-1]\frac{z}{c}\right]d\omega.$$

Assume that $f(z,t)$ is zero at $z=0$ for $t<0$. Then

$$f(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(0,t)\exp\left[i\omega t\right]dt\,,$$

is a function of ω analytic in the upper half-plane. So is also

$$\exp\left[i\omega[n(\omega)-1]\frac{z}{c}\right].$$

Therefore the product of these two functions is analytic in the upper half-plane and its Fourier transform must vanish for negative values of its argument τ. In (24) $\tau=t-z/c$, and therefore

$$f(z,t)=0 \qquad\qquad \text{for}\quad z>ct.$$

APPENDIX

Dispersion relations for the scattering of photons by electrons: the Thomas-Reiche-Kuhn sum rule.

In the paper by GELL-MANN, GOLDBERGER and THIRRING (GGT), the dispersion relations for the forward coherent scattering amplitude are given in the form

$$D_{\lambda\lambda}^{ii}(\omega')-D_{\lambda\lambda}^{ii}(0)=\frac{2\omega'^2}{\pi}\,P\int_{0}^{\infty}\frac{A_{\lambda\lambda}^{ii}(\omega)\,d\omega}{\omega(\omega^2-\omega'^2)}\,.$$

Here $D_{\lambda\lambda}^{ii}$ and $A_{\lambda\lambda}^{ii}$ are the real and imaginary parts of the scattering amplitude. In general the scattering amplitude is denoted by

$$D^{fi}_{\mu\lambda}(\boldsymbol{k}_f, \boldsymbol{k}_i) + iA^{fi}_{\mu\lambda}(\boldsymbol{k}_f, \boldsymbol{k}_i)\,,$$

signifying that the scatterer (*e.g.* an electron, bound or free) undergoes a transition from the state i (initial) to the state f (final), and the momentum and polarization of the photon change from $(\boldsymbol{k}_i, \lambda)$ to $(\boldsymbol{k}_f, \mu)$.

For forward coherent scattering $i \equiv f$, $\boldsymbol{k}_i \equiv \boldsymbol{k}_f$ $(\boldsymbol{k}_i| = |\boldsymbol{k}_f| = \omega$ in units $c = \hbar = 1)$ and $\lambda \equiv \mu$.

These transition amplitudes are derived from the quantum theory of fields. However, we can understand the formula classically just as well, by representing bound electrons as harmonic oscillators.

The expression for the field scattered by an oscillator can be found in elementary textbooks. It reads

$$a(\boldsymbol{e}^{(0)} + \boldsymbol{e}^{(1)}\cos\theta)\,\frac{\exp[i\omega(r-t)]}{r}\,,$$

where $\boldsymbol{e}^{(0)}$ and $\boldsymbol{e}^{(1)}$ are unit vectors, perpendicular to one another, $c = 1$ so that the wave number k is equal to the circular frequency ω, and

$$a = \frac{\omega^2\gamma}{4\pi} = a(\omega)$$

with

$$\gamma = \frac{e^2}{m}\,\frac{1}{\omega_0^2 - \omega^2 - i\omega\Gamma}$$

is the « scattering amplitude ». Thus $D^{ii}_{\lambda\lambda}$ and $A^{ii}_{\lambda\lambda}$ are essentially Re a and Im a. If Γ is regarded as constant, $a(\omega)$ does not tend to zero as $\omega \to \infty$. For this reason the GGT dispersion relations are satisfied not by $a(\omega)$ but rather by $(a(\omega) - a(0))/\omega$, a function of ω which does tend to zero as $\omega \to \infty$ and has no singularity for $\omega = 0$.

Further it can be shown that, denoting by σ the total cross section,

$$\sigma = \frac{4\pi}{\omega}\,\mathrm{Im}\,a\,. \qquad \text{(optical theorem).}$$

(Proof is straightforward: evaluate Poynting vector and flux through sphere etc.).

Therefore the GGT dispersion can be written

$$D^{ii}_{\lambda\lambda}(\omega') - D^{ii}_{\lambda\lambda}(0) = \frac{\omega'^2}{2\pi^2}\,P\!\int_0^\infty \frac{\sigma(\omega)\,\mathrm{d}\omega}{\omega^2 - \omega'^2}\,.$$

It is now $D^{ii}_{\lambda\lambda}(0) = 0$, and assuming that $D^{ii}_{\lambda\lambda}(\infty) = -e^2/m$ (*i.e.* for infinite energy of the photon the binding of the electron can be neglected, and $D^{ii}_{\lambda\lambda}(\omega)$ tends to its value — e^2/m for scattering by a free electron — in applying our classical formula notice that Γ must be constant for this to be true) we find, by going to the limit $\omega' \to \infty$.

$$\int_0^\infty \sigma(\omega)\,\mathrm{d}\omega = \frac{2\pi^2 e^2}{m}\,.$$

This is the Thomas-Reiche-Kuhn sum rule. Before GGT devised the above proof, it had been proved for dipole transitions ($\omega R \ll 1$) only.

Dispersion relations for non-relativistic potential scattering.

I shall now proceed to discuss dispersion relations in the modest problem of non-relativistic S-scattering by a central potential of short range.

As usual we define a new function $\varphi(r) = r\psi(r)$ from the Schrödinger wave function $\psi(r)$. Introducing suitable units, the equation obeyed by $\varphi(r)$ is

$$\text{(A.1)} \qquad \varphi'' - V\varphi = -i\,\frac{\partial\varphi}{\partial t}\,,$$

where the potential $V(r)$ vanishes for $r > r_0$.

All physically meaningful solutions must be zero for $r = 0$. We shall consider solutions of given energy $E = k^2$, which, for $r > r_0$, can always be written in the form

$$\text{(A.2)} \qquad \varphi = c\,(\exp[-ikr] - S(k)\exp[ikr])\exp[-iEt]\,. \qquad (r > r_0)\,.$$

In this formula $S(k)$ is the S-matrix (here an ordinary function of k) with the property

$$\text{(A.3)} \qquad S(k)\,S(-k) = 1\,.$$

Whenever k and E are real quantities, (A.2) represents a scattering wave function. In this case, $S(-k) = S^*(k)$ and therefore

$$\text{(A.4)} \qquad S(k)\,S^*(k) = 1\,,$$

which is the unitary condition for the S-matrix in the case of S-scattering.

Eq. (2) can be taken to hold when k and E are complex numbers. We write $k = k_1 + ik_2$, $k^* = k_1 - ik_2$ and $E = (k_1^2 - k_2^2) + 2ik_1k_2$. In this case eq. (A.3) is still valid while eq. (A.4) must be replaced by

$$\text{(A.4}') \qquad \frac{1}{S(k)} = [S(k^*)]^*\,,$$

reducing to (A.4) when k is real. We see from (A.4′) that the S-matrix is unitary only for real k. As for the physical interpretation of complex momenta, we note first that, if the momentum is purely imaginary ($k_1 = 0$), the energy is real and negative. Thus the wave function (A.2) would be that of a bound state if it were square integrable. This is the case if $S(k)$ has a zero on the negative imaginary axis. Then, for $r > r_0$, φ consists only of the first term $\exp[-ikr] = \exp[k_2 r]$, which is in fact square integrable for $k_2 < 0$. Note also that, by virtue of (A.3), there is a pole on the positive imaginary axis for every zero on the negative imaginary axis. These are the only poles of the function $S(k)$ in the upper half-plane, and they are all along the positive imaginary axis.

They are not, however, the only singularities in the upper half-plane. We can see this by studying the case of a square-well potential of range r_0 and depth $-V_0$, for which $S(k)$ can be given in an explicit form,

$$S(k) = \frac{1 + i\dfrac{k}{\sqrt{k^2+V_0}}\,\mathrm{tg}\,(r_0\sqrt{k^2+V_0})}{1 - i\dfrac{k}{\sqrt{k^2+V_0}}\,\mathrm{tg}\,(r_0\sqrt{k^2+V_0})}\exp[-2ir_0k]\,. \tag{A.5}$$

Clearly $S(k)$ has poles at values of k for which the denominator is zero (bound states), whereas the factor

$$\exp[-2ir_0k]\,,$$

has an essential singularity at infinity in the upper half-plane, since $\exp[2r_0k_2]$ tends to infinity as $k_2 \to \infty$ more rapidly than any power of k_2. The function

$$S(k)\exp[2ir_0k]\,,$$

on the other hand, has no essential singularity at infinity. For potentials of range r_0 other than the square-well potential, the essential singularitity would be of the type $\exp[-iak]$ with $a \leqslant 2r_0$. Thus it is killed by the factor $\exp[2ir_0k]$ in the upper half-plane, while an essential singularity is introduced in the lower half plane.

Let us for a moment forget about bound states. We can then say: the function $S(k)\exp[2ir_0k]$ is analytic in the upper half-plane. It would obey dispersion relations if its limit as $|k| \to \infty$ were zero. This, however, is not the case, as general studies of the S-matrix have shown. Consequently $S(k)\exp[2ir_0k]$ must be multiplied by a suitable convergence factor. This must be done in such a way that no new singularities are introduced. Therefore, if we multiply by the convergence factor $1/k$, say, we must subtract $S(0)/k$. We obtain the function

$$\frac{S(k)\exp[2ir_0k] - S(0)}{k}\,, \tag{A.6}$$

analytic in the upper half-plane and regular for $k = 0$. Writing dispersion relations for (A.6), and putting

$$\text{(A.7)}\quad \begin{cases} D(k) = \dfrac{1}{2k}\,\mathrm{Im}\,[S(k)\exp[2ir_0k]] = \mathrm{Re}\left[\dfrac{1}{2ik}\,S(k)\exp[2ir_0k]\right], \\[2ex] A(k) = -\dfrac{1}{2k}\,\mathrm{Re}\,[S(k)\exp[2ir_0k]] = \mathrm{Im}\left[\dfrac{1}{2ik}\,S(k)\exp[2ir_0k]\right], \end{cases}$$

(D and A are essentially the real and the imaginary parts of the first term of the scattering amplitude

$$f(\theta) = \frac{1}{2ik}\sum_l (2l+1)\,P_l(\cos\theta)\,(\exp[2i\eta_l]-1)$$

multiplied by $\exp[2ir_0k]$) we obtain

$$\text{(A.8)}\quad \left|\begin{aligned} A(k') &= -\frac{S(0)}{2k'} + \frac{2}{\pi}\,P\!\int_0^\infty \frac{k'D(k)}{k'^2-k^2}\,\mathrm{d}k\,, \\[2ex] D(k') &= \frac{2}{\pi}\,P\!\int_0^\infty \frac{kA(k)}{k^2-k'^2}\,\mathrm{d}k\,, \end{aligned}\right.$$

where the extra term occurs only in one of the equations since $S(0)=\pm 1$ and therefore $\mathrm{Im}\,S(0)=0$.

Further, we must take care of the bound states, if there are any. We stated at the beginning that, if a function $f(z)$ is analytic except at certain singular points $z_1, z_2 \ldots$, then the integral of $f(z)$ along a contour c enclosing these is not zero. However, if we add to the contour a series of small circles around the singularities, all described in the clock-wise direction if c is described in the anticlockwise direction, then the integral is again zero:

$$\int_{\underset{\curvearrowright}{c}} f(z)\,\mathrm{d}z + \left(\int_{\underset{\curvearrowleft}{c_1}} + \ldots + \int_{\underset{\curvearrowleft}{c_n}}\right) f(z)\,\mathrm{d}z = 0\,,$$

or:

$$\int_{\underset{\curvearrowright}{c}} = \int_{\underset{\curvearrowright}{c_1}} + \ldots + \int_{\underset{\curvearrowright}{c_n}}.$$

For this reason we have to add to the right member of the dispersion relation for $(S(k)\exp[2ir_0k]-S(0))/k$ the integrals

$$\text{(A.9)}\quad \frac{1}{\pi i}\,k'\sum_x \int_{c_x} \frac{S(k)\exp[2ir_0k]-S(0)}{k(k-k')}\,\mathrm{d}k\,,$$

(c_α around the pole k_α on the positive imaginary axis). Correspondingly the terms

$$\sum_\alpha \frac{k' b_\alpha}{k_\alpha^2(k_\alpha^2 - k'^2)} \qquad \text{and} \qquad -i\sum_\alpha \frac{b_\alpha}{k_\alpha(k'^2 - k_\alpha^2)},$$

must be added to the right members of the eqs. (A.8). Here

$$b_\alpha = \operatorname{Res}\left[kS(k)\exp[2ir_0k]\right]_{k=k_\alpha}. \qquad (\text{Res} = \text{residue}).$$

Problem. – Write dispersion relations for $S(k)/(k^2 - k_0^2)$, (k_0 real); in integrating along the real axis the points $+k_0$ and $-k_0$ must be avoided by means of indentations. Let $k_0 \to 0$ and obtain (A.8) [11]. The convergence factor $1/(k^2 - k_0^2)$ is employed by GOLDBERGER and others in deriving dispersion relations for pion-nucleon scattering.

Causality in non-relativistic potential scattering.

I now wish to show how the dispersion relations for $(S(k)\exp[2ikr_0] - S(0))/k$ can be derived from certain causality requirements in the time-dependent scattering theory.

For brevity's sake we shall assume that there are no bound states.

If the wave function $\varphi(r', t')$ is known at a time t', its values at a later time t are given by

$$\varphi(r, t) = \int_0^\infty K(r, t; r', t')\varphi(r', t')\,\mathrm{d}t', \tag{A.10}$$

where the propagation function $K(r, t; r', t')$ has the following properties:

$$\text{(A.11)}\quad \begin{cases} 1)\quad K(r, t; r', t') = \delta(r - r'), & \\ 2)\quad K(r, t; r', t') = 0, & \text{for } t < t', \\ 3)\quad \left(i\dfrac{\partial}{\partial t} + \dfrac{\partial^2}{\partial r^2} - V\right)K(r, t; r', t') = 0. & \text{for } t > t'. \end{cases}$$

Since there are no bound states, the function K is explicity given by

$$K(r, t; r', t') = \frac{1}{2\pi}\eta(t - t')\int_0^\infty \varphi_k(r)\varphi_k^*(r')\exp[-iE_k(t - t')]\,\mathrm{d}k, \tag{A.12}$$

where

$$\eta(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0, \end{cases}$$

and the functions $\varphi_k(r)$ are defined as in (A.2),

$$\varphi_k(r) = \exp[-ikr] - S(k)\exp[ikr]\,. \qquad r > r_0. \tag{A.13}$$

(The completeness relation for the functions $\varphi_k(r)$ is obtained by putting $t = t'$ in (A.12) and using the first of eqs. (A.11)).

We then have for $t > t'$ and $(r, r') > r_0$

$$K(r, t; r', t') = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\exp[-ik(r-r')] - S(k)\exp[ik(r-r')]\cdot \\ \cdot\exp[-iE_k(t-t')]\,\mathrm{d}k]\,. \tag{A.14}$$

In the absence of scattering ($V = 0$) $\varphi_k(r) = \exp[-ikr] - \exp[ikr]$ and the propagation function for $t > t'$ is

$$K_0(r, t; r', t') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[-ik(r-r')] - \exp[ik(r+r')]\cdot \\ \cdot\exp[-iE_k(t-t')]\,\mathrm{d}k\,. \tag{A.15}$$

Assume now that for $t < 0$ the incident wave packet $\varphi_i(r, t)$ has not yet reached the scatterer. Schützer and Tiomno's causality requirement [7] is that there must not be any scattered wave until the impinging wave has reached the scatterer. Thus, for $t < 0$ $\varphi(r, t) \equiv \varphi_i(r, t)$ and therefore

$$\varphi(r, t) = \varphi_i(r, t) + \varphi_s(r, t) = \int_0^{\infty} K(r, t; r', t')\,\varphi_i(r', t')\,\mathrm{d}r'. \qquad t' < t < 0; \tag{A.16}$$

On the other hand,

$$\varphi_i(r, t) = \int_0^{\infty} K_0(r, t; r', t')\,\varphi_i(r', t')\,\mathrm{d}r'. \tag{A.17}$$

Subtracting (A.17) from (A.16) we obtain ($t' < t < 0$)

$$\varphi_s(r, t) = \int_0^{\infty}[K(r, t; r', t') - K_0(r, t; r', t')]\,\varphi_i(r', t')\,\mathrm{d}r' = \tag{A.18}$$

$$= \frac{1}{2\pi}\int_0^{\infty}\mathrm{d}r'\left[\int_{-\infty}^{\infty}(1 - S(k))\exp[ik(r+r')]\exp[-iE_k(t-t')]\,\mathrm{d}k\right]\varphi_i(r', t')\,,$$

and making use once more of the requirement that the scattered wave must

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vanish for $r > r_0$ and $t' < t < 0$ we infer from (A.18) that

$$\int_{-\infty}^{\infty} [1 - S(k)] \exp[ik(r + r_0)] \exp[-iE_k t] \, \mathrm{d}k = 0 \,. \qquad (t > 0,\; r > r_0) . \tag{A.19}$$

This equation, as is easy to show, is valid also for $t = 0$, and can be written in the more significant form

$$\int_{-\infty}^{\infty} [1 - S(k)] \exp[2ikr_0] \exp[-ikz] \, \mathrm{d}k = 0, \qquad \text{for } z = r_0 - r < 0 \,, \tag{A.20}$$

showing that $S(k) \exp[2ikr_0]$ is the Fourier transform of a function which is zero for negative values of its argument z. By some manipulations it can be shown that the function $(S(k) \exp[2ikr_0] - S(0))/k$ has the same property. Since this latter function is square integrable, it must obey dispersion relations according to Titchmarsh's theorem.

There is a snag in all this. A wave packet which vanishes rigorously outside a certain region at all times $t < 0$ cannot be constructed in non-relativistic wave mechanics, since the energy is positive definite in this theory. It is possible in relativistic mechanics (energy $\pm\sqrt{k^2 + m^2}$) though there are difficulties here too (energies between $+m$ and $-m$ are unphysical).

REFERENCES

[1] B. Gross: *Phys. Rev.*, **59**, 748 (1941).
[2] H. W. Bode: *Network Analysis and Feedback Amplifier Design* (New York, 1945).
[3] R. Kronig: *Journ. Opt. Soc. Amer.*, **12**, 547 (1926).
[4] H. A. Kramers: *Atti Congresso Internazionale dei Fisici, Como*, **2**, 545 (1927).
[5] J. S. Toll: *Thesis* (Princeton, 1952).
[6] J. M. Jauch and F. Rohrlich: *Electrons and Photons* (Cambridge, Mass, 1955).
[7] W. Schützer and J. Tiomno: *Phys. Rev.*, **83**, 249 (1951).
[8] N. G. van Kampen: *Phys. Rev.*, **89**, 1072 (1953); **91**, 1267 (1953).
[9] M. Gell-Mann, M. L. Goldberger and W. E. Thirring: *Phys. Rev.*, **95**, 1612 (1954).
[10] E. C. Titchmarsh: *Fourier Integrals* (Oxford, 1937), p. 119.
[11] E. Corinaldesi: *Nucl. Phys.* **2**, 420 (1956).
[12] N. N. Bogoljubov: *Introduction to Quantum Field Theory* (Moscow, 1957).

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Dispersion Relations.

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1. – General remarks.

Whereas previous methods of field theory such as covariant perturbation theory still had some intuitive basis, it is a particular feature of dispersion theory that intuition is almost of no help. This is because analytic functions are fairly strange objects: for instance, changing slightly an analytic function on some interval of the real axis can completely alter its analytic properties.

Example. - The S-wave scattering matrix in an exponential potential exhibits « redundant poles » in the upper half plane; if however the potential is cut off at however large a distance one pleases, these poles disappear so that completely different analytic properties apply to two situations which are not physically distinguishable.

Such a situation arises from the sharpness of the causality requirement: for instance, requiring that a one dimensional wave packet

$$\int_{-\infty}^{\infty} \mathrm{d}k f(k) \exp\left[ik(x-t)\right],$$

vanishes for $x>0$ at $t=0$ implies that $f(k)$ is analytic in the upper half plane (Titschmarsh theorem). Then it follows that it vanishes for $x>t$ (applying Cauchy's theorem). Since the scattered wave can be written

$$\int_{-\infty}^{+\infty} \mathrm{d}k\, f(k) \exp\left[ik(x-t)+i\delta(k)\right],$$

(*) The following are notes taken by Drs R. STORA and B. UDGAONKAR. The Author is indebted to them for carrying out this unrewarding task.

the requirement that the scattered wave does not propagate faster than the velocity of light (*i.e.* vanishes for $x > t$) imposes that also $\delta(k)$ must be analytic in the upper half plane. This causality requirement is however a very detailed one which cannot be tested experimentally: indeed there exists a theorem due to Dyson which points to this practical impossibility.

Dyson's theorem: given two arbitrary functions $A(k)$, $D(k)$ and given an interval d and a small number ε, it is always possible to find functions $A'(k)$, $D'(k)$, satisfying dispersion relations, which are approximate to $A(k)$ and $D(k)$ in the interval d, with the accuracy ε, *i.e.*

$$|A(k)-A'(k)|<\varepsilon\,,\qquad |D(k)-D'(k)|<\varepsilon\,,\qquad \text{for } |k|<d\,.$$

Thus, let $A(k)$ and $D(k)$ be some experimental data, whose approximations $A'(k)$, $D'(k)$ fulfil dispersion relations in the interval d by virtue of some fast vanishing at infinity; this does not mean that $A(k)$ and $D(k)$ fulfil dispersion relations since a small change of $A'(k)$ and $D'(k)$ in the interval d may induce a large change outside d. For instance one may have the situation pictured here.

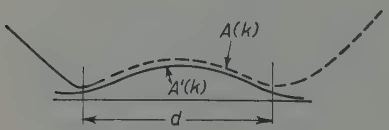

Fig. 1.

If on the other hand, dispersion relations are not fulfilled only using the experimental data in the interval d, one cannot say anything before one knows more data outside this interval (*e.g.* one may forget a bump outside d).

So that practical tests of dispersion relations are only valid in so far as the assumptions made on the behaviour of the functions involved outside the interval reached by experiments are valid.

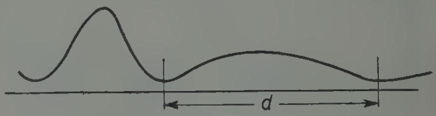

Fig. 2.

In some cases, some singularities in the upper half plane are allowed by a less stringent causality requirement, namely that of macroscopic causality. Consider for instance the wave packet

$$\int_{\infty}^{+\infty} \mathrm{d}\omega f(\omega)\exp[i(kx-\omega t)]\qquad \text{where}\qquad k=\sqrt{\omega^2-\mu^2}\,.$$

If $f(\omega)$ has poles in the upper half plane, there will be contributions of such poles to the scattered wave: for instance let $i\omega_0$ be such a pole of $f(\omega)$ which yields the contribution to the scattered wave:

$$\sim f(\omega_0)\exp\left[i\sqrt{-\omega_0^2-\mu^2}\,x+\omega_0 t\right]=f(\omega_0)\exp\left[-(\sqrt{\omega_0^2+\mu^2}-\omega_0)x-\omega_0(x-t)\right],$$

for $x > t$ this contribution decays exponentially and vanishes provided x is large enough, namely if one only looks at the asymptotic behaviour of the scattered wave; this essentially amounts to requiring macroscopic causality. Thus we see that macroscopic causality allows singularities which would not appear if microscopic causality were required.

Hence intuitively microscopic causality seems highly unphysical. Yet it turns out that local field theories imply microscopic causality; whether the converse is true or not is not known: as all the implications of microscopic causality are not known it is a very difficult question whether dispersion relations define local field theories or not; we shall not be concerned with such problems but rather with some restricted consequences of microscopic causality which can be used to connect experimental data.

2. – Microscopic causality in field theory.

Consider a field $\varphi(x)$; it is useful to introduce a coupling of this field with an external source giving rise to a Lagrangian

$$L' = \varphi(x)\varrho(x) .$$

Microscopic causality is implied in the following formula:

$$\langle f - \infty | \delta\varphi(x) | i - \alpha\rangle = i\int_{-\infty}^{x} \langle f - \infty | [\varphi(x), L(x')] | i, - \infty\rangle =$$

$$= i\int_{-\infty}^{x} \mathrm{d}x' \varrho(x') \langle f - \infty | [\varphi(x), \varphi(x')] | i - \infty\rangle ,$$

which gives the variation of the field (or rather its matrix elements between two states at time $-\infty$) due to the presence of the source.

If $[\varphi(x), \varphi(x')] = 0$ for x and x' separated by a space-like interval, then the variation of the field only occurs at point x in the future light cone of the point x where the source is. That $[\varphi(x), \varphi(x')] = 0$ for $x \sim x'$ was shown for free fields and should hold in general. Hence field theory requires that no signal can be exchanged between two points separated by a space like interval, and measurements at such points do not interfere. As here again we are faced with a « sharp edge » statement, we should expect some non-trivial analyticity properties. Whereas statements concerning fields at a point (which involve infinitely high frequencies) lead to a sharp causality requirement, one should not expect as much for particles which are low lying states of the system.

Indeed, a sharp localization is not possible since it would involve such high energies that other identical particles would be created, which are undistinguishable from the particle under observation. Never the less field theory allows us to relate the propagation of particles to microscopic causality.

The propagation of a particle is given by studying the amplitude for a particle being created at point x to be destroyed at point x': in particular, the amplitude for a state i at $-\infty$ to go into a state f at time $+\infty$ under the action of the source ϱ is for free particles given by:

$$\langle f+\infty | i-\infty\rangle \sim \int \mathrm{d}x\,\mathrm{d}x'\varrho(x)\varrho(x')\langle f|P(\varphi(x)\varphi(x'))|i\rangle =$$
$$= \int \mathrm{d}x\,\mathrm{d}x'\varrho(x)\varrho(x')\langle f|\theta(x-x')\varphi(x)\varphi(x') \pm \theta(x'-x)\varphi(x')\varphi(x)|i\rangle\,.$$

In particular, if $|i\rangle = |f\rangle = |0\rangle$ one has the amplitude for vacuum to stay vacuum in the presence of the external source, which is described by the propagator of the particle in vacuum: $\langle 0|P(\varphi(x)\varphi(x'))|0\rangle$ which does not vanish rigorously outside the light cone but damps out exponentially in a Compton wave length of the particle in the neighbourhood of the light cone. Although one does not get here a « sharp edge » statement, the content of such a description must be essentially equivalent to that in terms of fields as the chronological P-product acquires only an invariant meaning if $[\varphi(x), \varphi(x')] = 0$ for $x \sim x'$.

If the states $|i\rangle$ and $|f\rangle$ are one particle states, the function

$$\langle f|P(\varphi(x)\varphi(x'))|i\rangle\,,$$

essentially describes the propagation of the φ-particle in the presence of another particle which goes from state $|i\rangle$ to state $|f\rangle$ that is to say is closely connected with the amplitude of the corresponding scattering process.

3. – Characterization of states - Energy spectrum.

As one does not have any solvable relativistic theory in 4-dimensional space, one has to make a number of assumptions based on experience:

- the eigenvalues p_μ of the energy momentum operator P_μ are time like vectors ($P_\mu P^\mu = M^2 > 0$) in the forward light cone ($P^0 = E > 0$);
- there exists a vacuum state: $P_\mu|0\rangle = |0\rangle$;
- the mass spectrum is as shown in the diagram (Fig. 3).

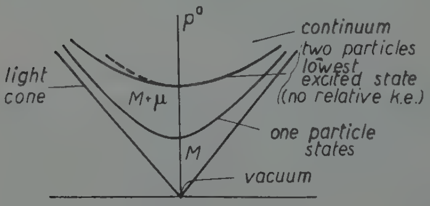

Fig. 3.

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We shall need in the following operators creating one particle states. They can easily be obtained by remarking that, by virtue of translation invariance, one has:

$$\tag{3.1} \begin{aligned} \langle p'|\varphi(x)|p\rangle &= \langle p'|\exp[iPx]\,\varphi(0)\exp[-iPx]|p\rangle\,, \\ &= \exp[i(p'-p)\cdot x]\langle p'|\varphi(0)|p\rangle\,. \end{aligned}$$

So, if one wants to produce an operator which has non vanishing matrix elements only between vacuum ($\varphi = 0$) and one particle states (φ' such that $p'^2 = m^2$), one has to select out of $\varphi(x)$ that part which has the right energy momentum dependence. This can be done either by integrating (3.1) with the proper exponential or alternatively by the following limiting procedure. From

$$\int \exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,\varphi(x)\,\mathrm{d}^3x\,,$$

select out the part whose time dependence is

$$\exp[-ix_0\sqrt{k^2+M^2}]\,,$$

by considering

$$\lim_{t\to\pm\infty}\int \mathrm{d}^3x\,\exp[-k\cdot x]\,\varphi(x)\,,$$

$$\left(\lim_{t\to-\infty} f(t) \sim \frac{1}{T}\int_{-2T}^{-T}\mathrm{d}t\,f(t)\right).$$

This procedure can be made relativistically invariant by writing:

$$\tag{3.2} |1_f^{\overset{\text{out}}{\text{in}}}\rangle = \lim_{\sigma\to\pm\infty}\int \mathrm{d}\sigma^\mu f^*(x)\,\overleftrightarrow{\partial^\mu}\varphi(x)\,|0\rangle = \varphi^{(-)}_{f\pm\infty}\,|0\rangle\,,$$

where $|1_f^{\overset{\text{out}}{\text{in}}}\rangle$ is an ingoing ($\sigma = -\infty$) or outgoing ($\sigma = +\infty$) one particle state with a positive energy wave function f, solution of the free wave equation (Klein-Gordon).

The standard notation $\overleftrightarrow{\partial^\mu}$ has the following meaning:

$$\tag{3.3} A\overleftrightarrow{\partial^\mu}B = A\left(\frac{\partial}{\partial x_\mu}B\right) - \left(\frac{\partial}{\partial x_\mu}A\right)B\,.$$

If f were a negative energy wave function, this state would have negative energy but since, by assumption we have only positive energy states, the right hand side of Eq. (3.2) vanishes in this case: $\varphi_f^{(+)}|0\rangle = 0$.

In order to construct many particle states, it is sufficient to take wave packets f which vanish outside some non-overlapping regions of space at times $\pm\infty$, for instance a two-particle state would be:

$$|1_f^{\text{in}} 1_{f'}^{\text{in}}\rangle = \varphi_{f,-\infty}^{(-)}\, \varphi_{f',-\infty}^{(-)}\,|0\rangle\,.$$

The transformation which allows us to go from ingoing states to outgoing states just defines the S-matrix. We shall apply these results to the pion nucleon scattering problem.

4. – Two-particle scattering.

4·1. *The scattering amplitude.* – We seek to express the amplitude for a meson of four-momentum k to be scattered from a nucleon of momentum p into a meson of momentum $\boldsymbol{k}'$, the final nucleon having momentum $\boldsymbol{p}'$:

$$\langle p' k'_{\text{out}}\,|\,p k_{\text{in}}\rangle = \langle p' k'_{\text{out}}\,|\int\limits_{\sigma=-\infty} \mathrm{d}\sigma^\mu f_k^*(x)\,\overleftrightarrow{\partial}_\mu\,\varphi(x)\,|\,p_{\text{in}}\rangle\,, \tag{4.1}$$

$f_k(x)$ is a plane wave of momentum k:

$$|p_{\text{in}}\rangle = |p_{\text{out}}\rangle = |p\rangle\,,$$

is a one nucleon state. By partial integration one has:

$$\begin{aligned}\langle p' k'_{\text{out}}\,|\int\limits_{\sigma=-\infty} \mathrm{d}\sigma^\mu f_k^*(x)\,\overleftrightarrow{\partial}^\mu\varphi(x)\,|\,p_{\text{in}}\rangle &= \langle p' k'_{\text{out}}\,|\int\limits_{\sigma=+\infty} \mathrm{d}\sigma^\mu f_k^*(x)\,\overleftrightarrow{\partial}_\mu\varphi(x)\,|\,p\rangle - \\ &\quad -\int \mathrm{d}^4x f_k^*(x)\,(\Box_x^2 - m^2)\,\langle p' k'_{\text{out}}\,|\,\varphi(x)\,|\,p\rangle\,,\end{aligned} \tag{4.2}$$

since $(\Box^2 - m^2) f_k(x) = 0$. (This is just Green's theorem applied to the functions $f_k(x)$, $\varphi(x)$ and the Klein-Gordon operator.)

Thus:

$$\langle p' k'_{\text{out}}\,|\,p k_{\text{in}}\rangle = \langle p' k'_{\text{out}}\,|\,k_{\text{out}} p_{\text{out}}\rangle - \int \mathrm{d}^4x f_k^*(x)\,(\Box_x^2 - m^2)\,\langle p' k'_{\text{out}}\,|\,\varphi(x)\,|\,p\rangle\,. \tag{4.3}$$

We apply this procedure a second time to take $\langle k'_{\text{out}}|$ out of the final state:

$$\begin{aligned}&\int \mathrm{d}^4x f_k^*(x)\,(\Box^2 - m^2)\int\limits_{\sigma'=+\infty} \mathrm{d}\sigma'^\mu f_{k'}(x')\,\langle p'\,|\,\overleftrightarrow{\partial}'^\mu\varphi(x')\varphi(x)\,|\,p\rangle = \\ &= \int \mathrm{d}^4x\,\mathrm{d}^4x' f_k^*(x) f_{k'}(x')\,(\Box_x^2 - m^2)\,(\Box_{x'}^2 - m^2)\,\langle p'\left|\begin{matrix} P(\varphi(x')\varphi(x)) \\ \theta(x-x')[\varphi(x')\varphi(x)]\end{matrix}\right|p\rangle\,.\end{aligned} \tag{4.4}$$

It is clear that the surface term at $\sigma' = -\infty$ vanishes as it destroys a meson $|k'_{\text{in}}\rangle$ in a state where there is only one nucleon.

Furthermore, one can at will write

$$(4.5) \qquad \varphi(x')\varphi(x)|_{x'=+\infty} = \left\{ \begin{array}{c} P(\varphi(x')\varphi(x)) \\ \text{or} \\ \theta(x'-x)[\varphi(x')\varphi(x)] \end{array} \right|_{x'=+\infty}$$

since $x' = +\infty >$ any x

$$\left(\begin{array}{ll} P\varphi(x')\varphi(x) = \theta(x'-x)\varphi(x')\varphi(x) + \theta(x-x')\varphi(x)\varphi(x')\,, & \\ \theta(x'-x) \begin{cases} = 1\,, & \\ = 0\,, & \end{cases} & \begin{array}{l} \text{if } x'^0 > x^0 \\ \text{if } x'^0 < x^0 \end{array} \end{array} \right)$$

Thus Eq. (4.4) is clear, and we can summarize:

$$(4.6) \qquad \langle p'k'_{\text{out}}|pk_{\text{in}}\rangle - \langle p'k_{\text{out}}|pk_{\text{out}}\rangle =$$
$$= \int \mathrm{d}^4x\,\mathrm{d}^4x' f^*_k(x) f_{k'}(x')(\Box^2_x - m^2)(\Box^2_{x'} - m^2)\langle p' \left| \begin{array}{c} p(\varphi(x)\varphi(x')) \\ \theta(x'-x)[\varphi(x')\varphi(x)] \end{array} \right| p\rangle\,.$$

It turns out that the Feynman amplitude (namely that which involves the P-bracket) does not have such convenient analytic properties as the retarded amplitude (namely that which is expressed in terms of the retarded commutator); indeed the latter is the Fourier transform of a function vanishing outside the forward light cone and we expect it to obey dispersion relations; from now on it is this one which we shall study:

$$(4.7) \qquad \langle p'k'_{\text{out}}|pk_{\text{in}}\rangle - \langle p'k'_{\text{out}}|pk_{\text{out}}\rangle =$$
$$= \int \mathrm{d}^4x\,\mathrm{d}^4x' \exp[i(kx - k'x')]\langle p'|R'\varphi(x')\varphi(x)|p\rangle\,,$$

where

$$(4.8) \qquad R'(\varphi(x')\varphi(x)) = (\Box^2_x - m^2)(\Box^2_{x'} - m^2)\theta(x'-x)[\varphi(x'), \varphi(x)]\,,$$

also vanishes outside the forward light cone.

Remark. One might think that for real particle momenta: $k^2 = k'^2 = m^2$, this amplitude vanishes identically because of the presence of the Klein-Gordon operators; this is not so however because the Fourier transform of

$$\langle p'|R'\varphi(x')\varphi(x)|p\rangle\,,$$

has singularities for $k^2 = k'^2 = m^2$.

We now take advantage of translation invariance in order to separate out the energy momentum conserving δ-function:

$$\langle p'|\varphi(x)\varphi(x')|p\rangle = \langle p'|\exp[iP\cdot x]\varphi(0)\exp[iP(x'-x)\varphi(0)\exp[-iPx']|p\rangle =$$
$$= \exp[i(p'x-px')]\langle p'|\varphi(0)\exp[iP\cdot(x'-x)]\varphi(0)|p\rangle\,,$$

the latter matrix element only depends on $(x'-x)$ and, using the new variables $(x+x')/2$, $x-x'$ one gets:

$$\text{(4.9)}\qquad \langle p'k'_{\text{out}}|pk_{\text{in}}\rangle - \langle p'k'_{\text{out}}|pk_{\text{out}}\rangle =$$
$$= (2\pi)^4\delta(p+k-p'-k')\int \mathrm{d}y\exp\left[\frac{i(k+k')y}{2}\right]\langle p'|R'\varphi\left(\frac{y}{2}\right)\varphi\left(-\frac{y}{2}\right)|p\rangle\,.$$

In the particular case of forward scattering where the spin and isotopic spin nucleon variables are specified by indices a and b, we write:

$$\text{(4.10)}\qquad M_{ab}(k) = \int \mathrm{d}y\exp[ik\cdot y]\langle p,a|R'\varphi\left(\frac{y}{2}\right)\varphi\left(-\frac{y}{2}\right)|p,b\rangle\,.$$

It is easy to show that

$$\text{(4.11)}\qquad M^*_{ab}(k) = M_{ba}(-k)\,,$$

i.e.

$$\text{(4.11)}\qquad M^+(k) = M(-k)\,.$$

This property allows us to define M for negative energies.

4·2. *Analytic properties of* $M(k)$. – Some technical difficulties arise in the discussion of the analytic properties of the scattering amplitude, because it is not as simple as a one-dimensional Fourier transform; so, in order to apply Titschmarch's theorem, we have to reduce the integral to a unidimensional one; we shall, for the time being, forget about degrees of freedom such as spin, isotopic spin, etc., which amounts to considering scalar neutral particles. In this case, the matrix element only depends on r^2, t; thus, one can perform right away the angular integration:

$$\text{(4.12)}\qquad M_{ab}(\omega) = i\int \mathrm{d}^4x\exp[ikx]\langle p,a|R'\varphi\left(\frac{x}{2}\right)\varphi\left(-\frac{x}{2}\right)|p,b\rangle =$$
$$= i\int \mathrm{d}r\, r\,\frac{\sin kr}{k}\int_r^\infty \mathrm{d}t\exp[i\omega t]\langle p,a|R'\varphi\left(\frac{x}{2}\right)\varphi\left(-\frac{x}{2}\right)|p,b\rangle\,.$$

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Notice that, although $k=\sqrt{\omega^2-m^2}$ has branch points at $\omega=\pm m$, $\sin kr/r$, which is even in k, does not.

Thus,

$$F(r,\omega)=ir\frac{\sin kr}{k}\int_r^{\infty}\exp[i\omega t]\,\mathrm{d}t\langle|R'\varphi\left(\frac{x}{2}\right)\varphi\left(-\frac{x}{2}\right)|\rangle, \tag{4.13}$$

is analytic in the upper half ω-plane, by Titschmarch's theorem. Also, it has the right boundedness properties for $\omega\to\infty$, since $k\sim\omega-m^2/2\omega$, and as $t>r$ implies that the part of $\sin(kr)$ which blows up is compensated by $\exp[i\omega t]$. This does not mean, however, that $M(\omega)=\int_r^{\infty}F(r,\omega)\,\mathrm{d}r$ is analytic. (For instance, although $\exp[-\alpha r]$ is an analytic function of α, $\int_0^{\infty}\exp[-\alpha r]\,\mathrm{d}r=1/\alpha$ has a pole.) For the time being, therefore, we only write dispersion relations for $F(r,\omega)$, assuming that it vanishes sufficiently fast for $\omega\to\infty$; otherwise, one has to make subtractions, as will be done; using the evenness property of F:

$$F(r,\omega)=F^*(r,-\omega)\,, \tag{4.14}$$

one obtains:

$$\operatorname{Re}F(r,\omega)-\operatorname{Re}F(r,\mu)=\frac{2(\omega^2-\mu^2)}{\pi}\,P\!\int_0^{\infty}\mathrm{d}\omega'\,\omega'\,\frac{\operatorname{Im}F(r,\omega')}{(\omega'^2-\mu^2)(\omega'^2-\omega^2)}\,. \tag{4.15}$$

(Better convergence has been insured by considering $F(r,\omega)/(\omega^2-\mu^2)$ instead of $F(r,\omega)$.)

In order to derive a dispersion relation for $M(\omega)$, one has to perform the r-integration and see whether the order of the r- and ω-integrations can be interchanged:

$$\begin{aligned}\operatorname{Re}M(\omega)-\operatorname{Re}M(\mu)&=\frac{2(\omega^2-\mu^2)}{\pi}\int_0^{\infty}\mathrm{d}r\int_0^{\infty}\mathrm{d}\omega'\,\omega'\,\frac{\operatorname{Im}F(r,\omega')}{(\omega'^2-\mu^2)(\omega'^2-\omega^2)}\overset{?}{=}\\&\overset{?}{=}\frac{2(\omega^2-\mu^2)}{\pi}\int_0^{\infty}\mathrm{d}\omega'\,\omega'\frac{\operatorname{Im}M(\omega')}{(\omega'^2-\mu^2)(\omega'^2-\omega^2)}\,.\end{aligned} \tag{4.16}$$

Initially, GOLDBERGER and coworkers interchanged this order of integration without too much care, and it turned out that the right answer is obtained in this fashion. We shall see right away that some non trivial problems come up in this connection. Let us rewrite $\operatorname{Im} F(r, \omega)$ as follows:

$$(4.17)\qquad \frac{1}{2i}\, r\, \frac{\sin kr}{k}\left[i\int\limits_0^\infty \mathrm{d}t \exp[i\omega t]\langle\,|R'\varphi\left(\frac{x}{2}\right)\varphi\left(-\frac{x}{2}\right)|\rangle - \right.$$

$$\left. -(-i)\int\limits_0^\infty \mathrm{d}t \exp[-i\omega t]\langle\,|R'\varphi\left(\frac{x}{2}\right)\varphi\left(-\frac{x}{2}\right)|\rangle^*\right],$$

changing variables in the second part, ($r \to -r$, $t \to -t$, which is trivial since only r^2 is involved), one gets:

$$(4.18)\qquad \operatorname{Im} F(r, \omega) =$$

$$= \frac{1}{2}\, r\, \frac{\sin kr}{k} \int\limits_{-\infty}^{+\infty} \mathrm{d}t \exp[i\omega t]\left\langle \left|\left[\overbrace{(\Box^2 - m^2)\varphi\left(\frac{x}{2}\right)}^{j(x/2)},\ \overbrace{(\Box^2 - m^2)\varphi\left(-\frac{x}{2}\right)}^{j(-x/2)}\right]\right|\right\rangle .$$

We can make explicit the space-time behaviour of the matrix element by using once more translation invariance:

$$(4.19)\qquad \left\langle p\left|\left[j\left(\frac{x}{2}\right), j\left(-\frac{x}{2}\right)\right]\right|p\right\rangle =$$

$$= \sum_n |\langle p|j(0)|n\rangle|^2 (\exp[i(p - p_n)x] - \exp[-i(p - p_n)x])\,,$$

and so, carrying out the time integration:

$$(4.20)\qquad \operatorname{Im} F(r, \omega) =$$

$$= \frac{1}{2}\, r\, \frac{\sin kr}{k} \sum_n \exp[i\boldsymbol{p}_n \cdot \boldsymbol{x}]\,|\langle p|j(0)|n\rangle|^2 [\delta(\omega - E_p + E_n) - \delta(\omega + E_p - E_n)]\,.$$

The intermediate states n which contribute to the sum contain at least one nucleon, since the number of nucleons is conserved. For $\omega > \mu$, $\delta(\omega - E_p + E_n) = 0$, since $E_n > E_p$, and, as k is real for $\omega > \mu$, $\operatorname{Im} F(r, \omega)$ is a well behaved function of r. There is however for $\omega < \mu$ a contribution due to a one nucleon state: $E_n = E_p + \omega$; then k is imaginary, and $\sin(kr)$ blows up exponentially for large r. Therefore one can interchange the order of r and ω integrations only for $\omega > \mu$.

Several methods have been proposed in order to circumvent the difficulty which occurs in the unphysical region ($|\omega| < \mu$). A first method due to BOGOLJUBOV consists in considering $M(\omega)$ as a function of m^2, which satisfies dis-

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persion relations for $m^2 < 0$; it is then proved that it can be analytically continued for $m^2 > 0$. This method involves however difficult mathematics in the theory of analytic functions of several complex variables. A second method also due to BOGOLJUBOV consists in writing dispersion relations for another function which has no singularity in the unphysical region (such a function is obtained by multiplying $F(r, \omega)$ by a polynomial which kills the troublesome singularity). One drawback of this method is that one cannot go back to $F(r, \omega)$ in a straightforward way since one does not know the residue at the pole $\omega = \omega_B$ (see later), so that one has to evaluate this residue directly. A third method due to GLASER amounts to considering $M(\omega)$ as an abelian integral by introducing a damping factor $\exp[-\alpha r]$ such that the interchange of the r- and ω-integrations is permissible for large enough α. It is then proved that $M_\alpha(\omega) = \int_0^\infty \exp[-\alpha r] F(r, \omega)\, \mathrm{d}r$ can be continued analytically down to $\alpha = 0$. Another different proof has been given by SYMANZIK.

The result of these considerations is that the interchange of the r- and ω-integrations is justified, so that one can write the following dispersion relation:

$$\text{(4.21)} \qquad \operatorname{Re} M(\omega) - \operatorname{Re} M(\mu) = \frac{2(\omega^2 - \mu^2)}{\pi} \int_0^\infty \mathrm{d}\omega'\omega' \frac{\operatorname{Im} M(\omega')}{(\omega'^2 - \mu^2)(\omega'^2 - \omega^2)} .$$

Using the optical theorem: $\operatorname{Im} M(\omega) = (4\pi/k)\,\sigma(\omega)$ this can be rewritten:

$$\text{(4.21)} \qquad \operatorname{Re} M(\omega) - \operatorname{Re} M(\mu) = \frac{\omega^2 - \mu^2}{2\pi^2} \int_0^\infty \mathrm{d}\omega' \frac{\sigma(\omega')}{(\omega'^2 - \mu^2)(\omega'^2 - \omega^2)} .$$

We shall later on evaluate the contribution to the dispersion integral due to the one nucleon state in the unphysical region. Since for massless particles this complication does not occur, we shall first study this case.

5. – Applications.

5'1. *Scattering of massless particles* (*photons*). – We have to introduce here complications due to spin and polarization. In the forward direction, the scattering amplitude for a photon of polarization e_λ into a photon of polarization e_μ ($K \cdot e_\lambda = K \cdot e_\mu = 0$) can be written as a matrix in the spin space of the target (we assume spin $\frac{1}{2}$ so that our considerations apply *e.g.* to Compton scattering from protons). We can write:

$$\text{(5.1)} \qquad M_{\lambda\mu}(\omega) = -i\int \mathrm{d}^4x \exp[ikx] \langle pa | R' A_\mu\left(\frac{x}{2}\right) A_\lambda\left(-\frac{x}{2}\right) | pb \rangle =$$
$$= \boldsymbol{e}_\mu \boldsymbol{e}_\lambda f_1(\omega) + i\boldsymbol{\sigma} \cdot \boldsymbol{e}_\mu \times \boldsymbol{e}_\lambda f_2(\omega) ,$$

(the amplitude must be a scalar under rotation and reflection). The i is usually introduced to make $i\boldsymbol{\sigma}\cdot\boldsymbol{e}_\mu\times\boldsymbol{e}_\lambda$ invariant under time reflections; then the evenness properties of $f_1(\omega)$ and $f_2(\omega)$ following from $M_{\lambda\mu}(\omega)=M^*_{\lambda\mu}(-\omega)$ are:

$$\text{(5.2)}\qquad \left\{\begin{aligned} f_1(\omega) &= d_1(\omega)+ia_1(\omega) = f_1^*(-\omega)\,, \\ f_2(\omega) &= d_2(\omega)+ia_2(\omega) = -f_2^*(-\omega)\,. \end{aligned}\right.$$

The dispersion relations are:

$$\text{(5.3)}\qquad \left\{\begin{aligned} d_1(\omega)-d_1(0) &= \frac{2\omega^2}{\pi}\int_0^\infty d\omega'\,\frac{a_1(\omega')}{\omega'(\omega'^2-\omega^2)}\,, \\ d_2(\omega)-\omega d_2'(0) &= \frac{2\omega^3}{\pi}\int_0^\infty d\omega'\,\frac{a_2(\omega')}{\omega'^2(\omega'^2-\omega^2)}\,. \end{aligned}\right.$$

It is more convenient to characterize the photons by circular polarization, so that the optical theorem allows to express $a_1(\omega')$ and $a_2(\omega')$ in terms of the cross sections for scattering of photons with spin parallel ($\uparrow\uparrow$) or antiparallel ($\uparrow\downarrow$) to the spin of the target:

$$\text{(5.4)}\qquad \left\{\begin{aligned} f_1(\omega) &= \tfrac{1}{2}[f_{\downarrow\downarrow}(\omega)+f_{\uparrow\downarrow}(\omega)]\,, \\ f_2(\omega) &= \tfrac{1}{2}[f_{\downarrow\downarrow}(\omega)-f_{\downarrow\uparrow}(\omega)]\,. \end{aligned}\right.$$

Correspondingly one has:

$$\text{(5.5)}\qquad \left\{\begin{aligned} a_1 &= \frac{4\pi}{\omega}\frac{1}{2}[\sigma_{\uparrow\uparrow}(\omega)+\sigma_{\uparrow\downarrow}(\omega)] = \frac{4\pi}{\omega}\,\sigma_{\text{unpolarized}}(\omega)\,, \\ a_2 &= \frac{4\pi}{\omega}\frac{1}{2}[(\sigma_{\uparrow\uparrow}\omega)-\sigma_{\uparrow\downarrow}(\omega)]\,. \end{aligned}\right.$$

In the case where the target is the proton then the cross sections are well approximated by the photoproduction cross sections. The dispersion relations can be then exploited because it can be proved that the dispersive parts of the amplitude at zero energy are expressible in terms of the charge and anomalous magnetic moments of the target as follows:

$$\text{(5.6)}\qquad \left\{\begin{aligned} d_1(0) &= -e^2/M\,, \\ d_2'(0) &= -2\mu^2_{\text{anom}}\,. \end{aligned}\right.$$

The first dispersion relation:

$$\text{(5.7)}\qquad d_1(\omega) = -\frac{e^2}{M}+\frac{\omega^2}{2\pi^2}\int_0^\infty d\omega'\,\frac{\sigma(\omega')}{\omega'^2-\omega^2}\,,$$

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is valid for both spin 0 and spin $\frac{1}{2}$ targets, and one may expect it to be valid for any spin (σ = cross section for unpolarized target). The first term corresponds to zero energy scattering whereas the integral term reminds of Rayleigh scattering, that is to say takes into account the polarizability of the scatterer.

In the case of Compton scattering from protons, the integral extends from μ to ∞, if one neglects terms of order e^4 or higher. Below meson photoproduction threshold $\omega'^2 - \omega^2 > 0$ so that there is a destructive interference between Thompson scattering and Rayleigh scattering, in the forward direction:

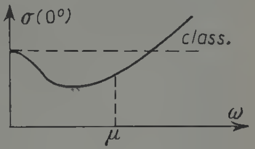

Fig. 4.

Though one would think that this decrease of the cross section makes the experimental situation more difficult, it is not so because there results a constructive interference in the backward direction. Such an effet is indeed very strong around the resonance ($\sigma_{\text{photoprod}} \sim e^2 f^2/\mu^2 \sim \sim e^2/M$) where most of the photoproduction cross section is due to a magnetic dipole transition.

It is of some interest to apply eq. (4.7) to nuclei; in this case

$$d_1(0) = -(Ze)^2/AM\,.$$

We try to use (5.7) for infinite photon frequency as one expects that binding effects become negligible and

$$d_{\text{nucleus}}(\infty) = Zd_{\text{proton}}(\infty) + Nd_{\text{neutron}}(\infty)\,. \tag{5.8}$$

One has:

$$\left\{\begin{aligned} d_{\text{proton}}(\infty) &= -\frac{e^2}{M} - \frac{1}{2\pi^2}\int_{\mu}^{\infty} \mathrm{d}\omega'\,\sigma_{\mathrm{p}}(\omega')\,,\\ d_{\text{neutron}}(\infty) &= -\frac{1}{2\pi^2}\int_{\mu}^{\infty} \mathrm{d}\omega'\sigma_{\mathrm{n}}(\omega')\,,\\ d_{\text{nucleus}}(\infty) &= -\frac{(Ze)^2}{AM} - \frac{1}{2\pi^2}\left[\mathrm{d}\omega'\sigma_{\text{production}}(\omega') + \int_{\mu}^{\infty}\mathrm{d}\omega'\sigma_{\text{absorption}}(\omega')\right]. \end{aligned}\right. \tag{5.9}$$

In the last formula, we had to include the possibility of photodisintegration below production threshold.

Thus one obtains:

$$\frac{1}{2\pi^2}\int_0^{\mu}\mathrm{d}\omega'\sigma_{\text{absorpt}}(\omega') = \frac{ZNe^2}{MA} + \frac{1}{2\pi^2}\int_{\mu}^{\infty}\mathrm{d}\omega'(\sigma_{\text{prod.}}(\omega') - Z\sigma_{\mathrm{p}}(\omega') - N\sigma_{\mathrm{n}}(\omega'))\,,$$

in order to express the reduction of meson production by binding effects we write:

$$\frac{1}{2\pi^2}\int_0^{\mu}\sigma_{\text{absorpt.}}(\omega') = \frac{ZNe^2}{MA} + \frac{1}{2\pi^2}\int_{\mu}^{\infty}\mathrm{d}\omega'\sigma_{\text{prod.}}(\omega')\,(1-R)\,, \tag{5.10}$$

which is a generalization of the sum rule for nuclei, in wich exchange effects are contained in the right hand side integral.

An estimate of these formula yields:

$$\frac{1}{2\pi^2}\int_0^{\mu}\sigma_{\text{absorpt.}}(\omega')\,\mathrm{d}\omega' = \frac{e^2}{M}\,\frac{ZN}{A}\left(1+0.1\,\frac{A^2}{NZ}\right); \tag{5.11}$$

the second term in the bracket which amounts to about 20% represents the effect of virtual mesons which manifests itself in the rise of the photodisintegration cross section at energies of the order of the meson rest mass.

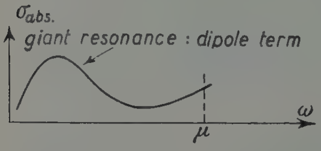

Fig. 5.

Before closing up this section we would like to make a remark on the sign of the zero energy forward scattering amplitude; though «sharp edge» causality stems from field theory, one would like to be sure that no group velocity larger than the velocity of light comes out of such a theory. The knowledge of the dispersive part of the amplitude is equivalent to that of the index of refraction from which the group velocity can be obtained; it turns out that $-e^2/M$ just measures the deviation of the latter from the velocity of light so that it had better be negative. If one were able to find theories in which this would be contradicted, the situation would become quite uncomfortable.

Though the optical theorem imposes the sign of the absorptive amplitude, there is no general argument to fix that of the dispersive part and it may be that the requirement that group velocities should be smaller than the velocity of light would give restrictions on the sign.

5·2. *π-meson-nucleon scattering.* – In the forward direction, the spin flip amplitude vanishes $\boldsymbol{k}\times\boldsymbol{k}'=0$ for $\boldsymbol{k}=\boldsymbol{k}'$ but isotopic spin degrees of freedom are still left. We write the amplitude as a matrix in the nucleon isotopic space:

$$M_{\alpha\beta}(\omega) = i\int \exp[ikx]\langle p\,|\,R'\varphi_\alpha\left(\frac{x}{2}\right)\varphi_\beta\left(-\frac{x}{2}\right)|\,p\rangle\,, \tag{5.12}$$

where α, β are the meson isotopic spin indices.

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By virtue of charge independence, one may write:

$$M_{\alpha\beta}(\omega) = T_1(\omega)\delta_{\alpha\beta} + T_2(\omega)[\tau_\alpha, \tau_\beta]\,. \tag{5.13}$$

Since $M_{\alpha\beta}(\omega) = M^+_{\alpha\beta}(-\omega)$ one has the evenness properties

$$\left\{\begin{aligned} T_1(\omega) &= T_1^*(-\omega)\,, \\ T_s &= D_s + iA_s\,, \\ T_2(\omega) &= -T_2^*(\omega)\,. \end{aligned}\right. \qquad (s = 1, 2): \tag{5.14}$$

These amplitudes can be at wish expressed in terms of those pertaining to a given isotopic spin, or in terms of the π^+ p, π^- p amplitudes:

$$T_1(\omega) = T_{\frac{1}{2}}(\omega) + \ T_{\frac{3}{2}}(\omega) = \tfrac{1}{2}\left(T_+(\omega) + T_-(\omega)\right),$$

$$T_2(\omega) = T_{\frac{1}{2}}(\omega) - 2\,T_{\frac{3}{2}}(\omega) = \tfrac{1}{2}\left(-T_+(\omega) + T_-(\omega)\right).$$

Since $T_1(\omega)$ and $T_2(\omega)$ have simple evenness properties they obey simple dispersion relations.

$$\left\{\begin{aligned} D_1(\omega) - D_1(\mu) &= \frac{1}{2}[D_-(\omega) + D_+(\omega)] - \frac{1}{2}[D_-(\mu) + D_+(\mu)] = \\ &= \frac{2(\omega^2 - \mu^2)}{\pi}\int_0^\infty \mathrm{d}\omega'\omega' \cdot \frac{1}{2}\,\frac{A_+(\omega') + A_-(\omega')}{(\omega'^2 - \omega^2)(\omega'^2 - \mu^2)}\,, \\ D_2(\omega) - D_2(\mu) &= \frac{1}{2}[D_-(\omega) - D_+(\omega)] - \frac{\omega}{2}[D'_-(\mu) - D'_+(\mu)] = \\ &= \frac{2\omega(\omega^2 - \mu^2)}{\pi}\int_0^\infty \mathrm{d}\omega'\,\frac{1}{2}\,\frac{A_-(\omega') - A_+(\omega')}{(\omega'^2 - \omega^2)(\omega'^2 - \mu^2)}\,. \end{aligned}\right. \tag{5.15}$$

The integrals at the right hand side contain distributions from the unphysical region which we shall now evaluate: we already saw that the absorptive part can be written as a sum over states.

$$A(\omega) = \sum_{\substack{n \\ p_n = k}} |\langle p|j|n\rangle|^2[\delta(E_p - E_n + \omega) - \delta(E_p - E_n - \omega)]\,. \tag{5.16}$$

For $0 < \omega < \mu$ one has a contribution from a one nucleon state such that

$$(5.17)\qquad \begin{cases} M - E_n - \omega = 0\,, & (E_p = M)\,, \\ E_n = \sqrt{k^2 + \mu^2} = M - \dfrac{\mu^2}{2M}\,, & (k^2 < 0)\,, \\ \omega \;= \omega_B = \dfrac{\mu^2}{2M}\,. & \end{cases}$$

Thus, the contribution to the dispersion integral stems from an absorptive part $|\langle p|j|n\rangle|^2 \delta(\omega - \omega_B) \cdot \langle p|j|n\rangle$ is related to the renormalized coupling constant which is involved in all low energy π-meson processes in the same way as $\langle p|j_0|p\rangle = e$ defines the renormalized charge in electrodynamics.

The pseudoscalar nature of this current implies a k^2 dependence

$$(5.18)\qquad \delta(\omega - \omega_B)|\langle p|j|p_n|^2 = 2\pi g^2 \frac{k^2}{(2M)^2}\,\delta(\omega - \omega_B) = 2\pi f^2 \left(\frac{k}{\mu}\right)^2 \delta(\omega - \omega_B)\,,$$

where

$$f^2 = g^2 \left(\frac{\mu}{2M}\right)^2.$$

Notice that this «bound state» term arises from $A^+(\omega)$ (and would arise from A^- for $\omega < 0$).

The final form of dispersion relations is thus:

$$(5.19)\qquad \frac{1}{2}[D_-(k) + D_+(k)] - \frac{1}{2}[D_-(0) + D_+(0)] = \frac{f^2}{M}\,\frac{k^2}{\omega^2 - (\mu/2M)^2} + \frac{k^2}{4\pi^2}\int_0^\infty \mathrm{d}k'\,\frac{\sigma_+(k') + \sigma_-(k')}{k'^2 - k^2}\,.$$

One interesting application of these relations is that made by U. HABERSCHAIM to determine the coupling constant. He uses the modified form:

$$\omega\,\mathrm{Re}\,T_2(\omega) - \frac{\omega^4}{2\pi^2}\int_\mu^\infty \mathrm{d}\omega'\,\frac{\sigma_-(\omega') - \sigma_+(\omega')k'}{2\omega'^2(\omega'^2 - \omega^2)} = 2f^2 + \frac{\omega^2}{2\pi^2}\int_\mu^\infty \frac{\sigma_-(\omega') - \sigma_+(\omega')}{2}\,\frac{k'\,\mathrm{d}\omega'}{\omega'^2}\,.$$

The left hand side is known and plotted against ω^2; when extrapolated to $\omega = 0$ this curve yields $f^2 \simeq 0.1$.

Another theoretical application is the derivation of the static model Low equations by assuming P-state scattering only; also some information is obtained on the S phase shifts which cannot be got from the static model.

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Fit to Dispersion Relations of Pion-Nucleon Scattering.

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1. – Introduction.

The pion-nucleon scattering is completely defined by the 4 amplitudes for flipping or non-flipping of the spin and isotopic spin of the nucleon. We may express these in terms of the scattering amplitudes in pure isotopic spin states as follows:

$$T_{\mathrm{N}} = \tfrac{1}{3}(T_{\frac{1}{2}} + 2\,T_{\frac{3}{2}}) \qquad \text{no flip of nucleon isotopic spin},$$

$$T_{\mathrm{F}} = \tfrac{1}{3}(T_{\frac{1}{2}} - \;\; T_{\frac{3}{2}}) \qquad \text{flip of nucleon isotopic spin}.$$

The amplitudes in the state of isotopic spin T can be expressed in terms of the phase shifts:

$$T = \frac{1}{2ik} \sum_l \left[l(\exp[2i\delta^-_{l,T}] - 1) + (l+1)(\exp[2i\delta^+_{l,T}] - 1)\right] P_l(\cos\theta)\,,$$

no flip of nucleon spin,

$$T = \frac{1}{2ik} \sum_l l(l+1)(\exp[2i\delta^-_{l,T}] - \exp[2i\delta^+_{l,T}]) \sin\theta\, P'_l(\cos\theta)\,,$$

flip of nucleon spin:

The phase shift $\delta^{\pm}_{l,T}$ refers to the state of isotopic spin T, relative orbital angular momentum l and total angular momentum $l \pm \frac{1}{2}$. Our 4 amplitudes are exhibited in the following table:

	spin-flip	isotopic spin-flip
N_{N}	no	no
N_{F}	no	yes
F_{N}	yes	no
F_{F}	yes	yes

These amplitudes treat the spin and isotopic spin in a completely symmetrical way. The relative magnitude of the amplitudes depends on the pion nucleon force. This force is both spin and isotopic spin dependent, so that all 4 will in general differ from zero.

Any complete dynamical theory of the pion-system must allow exact calculation of the four amplitudes defined above. However, we do not have a complete dynamical theory, but only certain approximate methods of treating the system. Dispersion theory does not constitute a complete dynamical theory, and therefore cannot be used to calculate the scattering amplitudes. It gives, from very general assumptions, a relation between the real and imaginary parts of each of the scattering amplitudes. These basic assumptions are:

a) *Causality.* Each of the scattering amplitudes is a complex function $S(\omega) = D(\omega) + iA(\omega)$ of the meson energy ω. S is defined physically for real values of ω greater than the meson mass μ. The causality condition then guarantees the existence of a function $M(z)$ of the complex variable z, which is analytic in the upper half z-plane and which reduces to $S(\omega)$ for z real and between μ and $+\infty$. The dispersion relations then follow from the analyticity properties of $M(z)$ (cfr. lecture of THIRRING).

b) *Unitarity condition*, or its consequence, the optical theorem

$$\sigma_{(\text{total})} = \frac{4\pi}{k}\,\text{Im}\,S(0)\,,$$

when $S(0)$ is the amplitude in the forward direction.

c) *Crossing theorem*, which allows us to express $M(\omega)$ for negative ω in terms of its value for positive ω:

$$M_{\pm}(\omega) = M^{*}_{\mp}(-\,\omega)\,.$$

In this case, the theorem is equivalent to the simultaneous operations of charge conjugation and time reversal.

d) *High energy behaviour of* $\sigma(\omega)$. It is necessary that the integral

$$\int\limits_{\mu}^{\infty}\sigma_{\text{tot}}(\omega)\,\omega^{-2}\,\mathrm{d}\omega\,,$$

converge.

e) *Behaviour of* $S(\omega)$ *in the unphysical region*, $0 < \omega < \mu$. S is assumed to differ from zero in this region only at the point $\omega = \mu^2/2M$ which corresponds to the physical nucleon state, and at which S has a singularity of

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the form

$$\operatorname{Im} S(\omega) = 2\pi f^2 \delta\left(\omega - \frac{\mu^2}{2M}\right) \qquad 0 < \omega < \mu .$$

The calculation of this expression for S involves a knowledge of the pion-nucleon coupling, and was made using the pseudo-scalar coupling. We do not discuss the derivation of the dispersion relation, but proceed immediately to the applications.

2. – **Sign ambiguity of phase shifts** [1].

In this work dispersion relations are used in the form given by GOLDBERGER, MIYAZAWA and OEHME:

$$(1)\qquad D_\pm(k) = \frac{1}{2}\left(1 + \frac{\omega}{\mu}\right) D_\pm(0) + \frac{1}{2}\left(1 - \frac{\omega}{\mu}\right) D_\mp(0) + \\ + \frac{k^2}{4\pi}\int_\mu^\infty \frac{\mathrm{d}\omega'}{k'}\left[\frac{\sigma_\pm(\omega')}{\omega' - \omega} + \frac{\sigma_\mp(\omega')}{\omega' + \omega}\right] \pm \frac{2f^2}{\mu^2}\,\frac{k^2}{\omega \mp \mu^2/2M} .$$

Here, $D_\pm(k)$ is the real part of the forward scattering amplitude for $\pi^\pm$-mesons in the laboratory system. D may be expressed in terms of the phase shifts as follows

$$D_+ = D_3 , \qquad D_- = \tfrac{1}{3}(3D_1 + D_3) ,$$

$$D_3 = \frac{k}{2k_b^2}\,(\sin 2\alpha_3 + 2 \sin 2\alpha_{33} + \sin 2\alpha_{31} + \dots) ,$$

$$D_1 = \frac{k}{2k_b^2}\,(\sin 2\alpha_1 + 2 \sin 2\alpha_{31} + \sin 2\alpha_{11} + \dots) ,$$

k and k_b are the pion momenta in the center of mass and laboratory system. Now, if we change the sign of all the α's, the cross section does not change, so that this sign cannot be fixed from a knowledge of $\mathrm{d}\sigma/\mathrm{d}\Omega$. However, the quantity D changes sign if the α's are changed. This allows us to use the dispersion relation to fix the sign. The right hand side of the relation (1) may be computed from the experimental value of $\sigma(\omega)$ and the value of $2f^2 = 0.161$. This may then be compared with the true possibilities for the left side, using the values of $D_\pm(0)$ computed from Orear's values

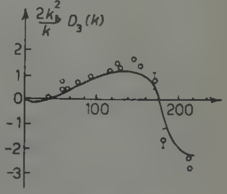

Fig. 1.

of the zero kinetic energy scattering lengths, $a_3 = -0.11$, $a_1 = +0.16$. It is found that the two quantities agree only for the choice of sign in which the large phase shift α_{33} is positive. This choice of sign also agrees with the sign determined from experiments on Coulomb interference in the π^+ scattering. We may therefore say that, in a rough way, the agreement is a test of the validity of dispersion relations (see Fig. 1).

3. – Fermi-Yang ambiguity.

There is another transformation of the phase shifts, discovered by YANG, which leaves the cross sections invariant. It does not leave the polarization unchanged. Therefore, to decide between the YANG phase shifts and the original set found by FERMI, either an actual or a theoretical experiment on the polarization of the final nucleon is necessary. Such a theoretical experiment was done in a paper by DAVIDON and GOLDBERGER [2] by means of the dispersion relation for the spin-flip amplitude, which is the amplitude containing information about the polarization. They used an especially convenient form of the dispersion relation which can be obtained from the usual form by algebraic modifications:

$$f^2 + Cx = \tfrac{1}{2}x\left[\operatorname{Re} a_3 - \omega\left(I_3(\omega) + \tfrac{1}{3}I_3(-\omega)\right)\right],$$

with

$$x = \frac{\omega}{1 + (\mu/2M)/\omega}, \qquad C = \frac{1}{3\pi}\int_1^\infty \mathrm{d}\omega' \, \frac{\operatorname{Im} a_3(\omega')}{\omega'}, \tag{2}$$

$$I_3(\omega) = \frac{1}{\pi} P\int_1^\infty \frac{\mathrm{d}\omega'}{\omega' - \omega}\,\frac{\operatorname{Im} a_3}{\omega'}.$$

Here, a_3 is the derivative with respect to $\sin\theta$ of the amplitude for isotopic spin $\frac{3}{2}$, evaluated in the forward direction. The simplification of neglecting the small $T = \frac{1}{2}$ amplitude has been made. Thus the derivative of the spin-flip amplitude becomes $-\frac{1}{3}a_3$. In terms of the phase shifts,

$$a_3 = \sum_{l=1}^\infty \frac{l(l+1)}{2i}\,\frac{1}{\eta^3}\left(\exp\left[2i\delta^+_{l\frac{3}{2}}\right] - \exp\left[2i\delta^-_{l,\frac{3}{2}}\right]\right).$$

The quantity C may be calculated from the experimental value of the phase shifts. However, since the integral converges rather slowly, the calculation is not so accurate, and therefore C is left as a parameter in the experimental

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fit; $I_3(\omega)$ converges more rapidly due to the additional power of ω in the denominator, so that its value may be more easily calculated. The comparison is then made by plotting the right hand side as a function of the quantity x. This must give a straight line with intercept f^2. The results for the two sets of phases are shown in the Fig. 2. The Yang phases must be excluded, since they give neither a straight line nor a positive value of f^2. The Fermi phases fall rather close to a straight line whose slope is consistent with the experimentally computed value of C.

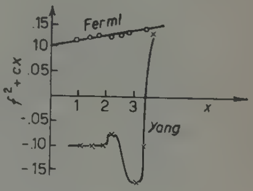

Fig. 2.

4. – Determination of the coupling constant f^2.

In order to calculate f^2, HABER-SCHAIM [3] has applied the dispersion relation for the isotopic spin-flip amplitude $T(\omega)$. The relation, rearranged for suitable comparison with data, is as follows:

$$\omega \operatorname{Re} T(\omega) - \frac{\omega^4}{2\pi^2}\int_\mu^\infty \left(\frac{\sigma_-(\omega') - \sigma_+(\omega')}{2}\right)\frac{k'\,\mathrm{d}\omega'}{\omega'^2(\omega'^2 - \omega^2)} =$$

$$= 2f^2 + \omega^2\left(\frac{1}{2\pi^2}\int_\mu^\infty \frac{\sigma_-(\omega') - \sigma_+(\omega')}{2}\,\frac{k'\,\mathrm{d}\omega'}{\omega'}\right) = 2f^2 + C\omega^2 .$$

Thus, the left side, plotted against ω^2, must give a straight line with intercept $2f^2$. For the calculation, HABER-SCHAIM chose two points, $\omega = 1$ and $\omega = 2$ where the experimental data are believed to be most reliable.

The point at $\omega = 1$ depends on the low energy behaviour of $\alpha_3 - \alpha_1$ and that at $\omega = 2$ near the resonance, depends mainly on the large α_{33} phase shift. These are the main pieces of experimental information used. The value $f^2 = 0.082 \pm 0.015$ was obtained. The error quoted is essentially that of $\alpha_3 - \alpha_1$ and is only significant as an order of magnitude. The slope of the straight line is consistent with the value of C obtained by direct calculation.

It is worth emphasizing that a mixture of π^+ and π^- data enters into this calculation.

5. – Work of Puppi and Stanghellini [4].

These authors applied the ordinary form (1) of the dispersion relations in order to work directly with the experimental angular distributions without making a phase shift decomposition. In the forward direction, D can be calculated from the coefficients a, b, c, of the angular distribution and from the total cross section using the optical theorem:

$$|f(\theta)|^2 = a + b\cos\theta + c\cos^2\theta\,,$$

$$|f(0)|^2 = |D|^2 + |A|^2 = a + b + c\,, \qquad A = \frac{k}{4\pi}\,\sigma_{\text{tot}}\,,$$

and

$$D = \sqrt{(a + b + c) - \frac{k^2}{(4\pi)^2}\,\sigma^2_{\text{tot}}}\,.$$

The results are shown in Figs. 3 and 4.

It is seen that the π^+ data fit quite well with $f^2 \approx 0.08$, both above and below the resonance. The π^- data however do not agree with the dispersion relation curve with $f^2 \approx 0.08$. It is necessary to use $f^2 \approx 0.04$ to obtain agreement up to resonance. We must, however, take this discrepancy with caution for several reasons: (i) π^- data are not as good as the π^+ data; (ii) there are small corrections to the dispersion relations, due to such things as the π^0-$\pi^\pm$ mass difference; (iii) errors are larger than the one shown, because errors of the total cross section have not been included.

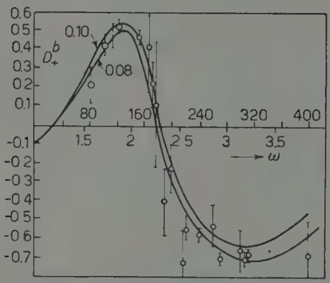

Fig. 3.

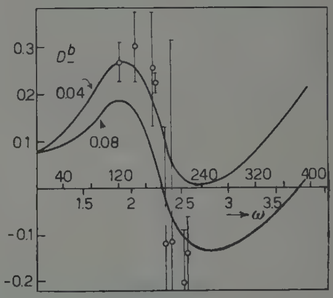

Fig. 4.

6. – Further work.

There has been a good deal of work in the last 2 years, by several people, with the aim of solving the discrepancy pointed out just now.

One point seems to be clear now, through the work of SALZMAN and BERTOCCHI-LENDINARA (Fig. 5); that the discrepancy is not so big as shown in the Fig. 4. The improvement comes essentially from the use of better π^- cross-

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section in the low energies region thanks to the recent experimental work at Liverpool and Rochester (CERN Conference, 1958). Further improvements will be possible as soon as new experimental information will be available. As HAMILTON and LOMON-ZAIDI have pointed out, the dispersion integrals are in fact very sensitive not only to the value at the cross-sections but also to their slope; therefore also small changes can have an effect.

There is also the problem of estimating correctly the errors in the analysis. The errors quoted in Fig. 4 come only from the experimental errors in the quantity D, but, as Hamilton has shown, the errors in the other terms of the relations are not negligible and taking them properly into account, the discrepancy tends to be reduced.

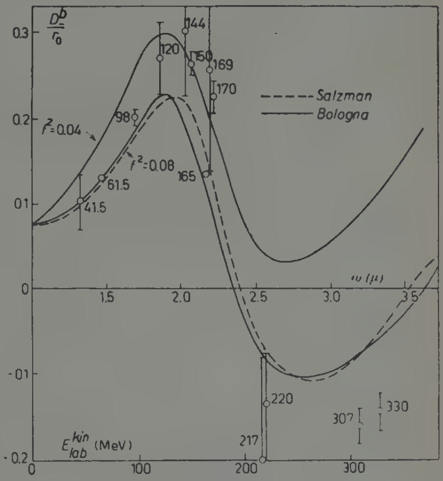

Fig. 5.

Finally there is the problem of the contribution of partial waves higher than s and p. Althout the situation is not yet clear, it is possible to say that their inclusion in the analysis will alter substantially the situation above the resonance, expecially in the relation for D_-.

In conclusion, although, the fit is not yet satisfactory and the discrepancy has not yet been removed, there is still the possibility of improvement.

Clearly this improvement could come only from better experimental results mainly for π^- cross-section.

REFERENCES

[1] H. L. ANDERSON, W. C. DAVIDON, U. E. KRUSE: *Phys., Rev.* **100**, 339 (1955).
[2] W. C. DAVIDON and M. L. GOLDBERGER: *Phys. Rev*, **104**, 1119 (1956).
[3] U. HABER-SCHAIM: *Phys. Rev.*, **104**, 1113 (1956).
[4] G. PUPPI and A. STANGHELLINI: *Nuovo Cimento*, **5**, 1305 (1957).

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Nucleon Structure.

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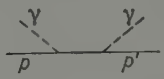
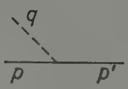
The problem of nucleon structure is becoming a very wide chapter of high energy physics. Scattering of photons may be a suitable way of investigating the structure of charge distribution. It is well known that scattering of light by atoms is structure dependent if the frequency is not too small.

If very hard γ-rays are scattered by protons, many virtual states, containing one or more mesons, play a very important role in the process. We cannot deal with the problem simply by considering interaction between a physical photon and a physical proton, represented by a graph of the type

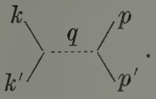


because in the intermediate state we cannot neglect complicated processes. A more suitable way of handling the problem is to consider situations in which the physical nucleon simply absorbs the photon and changes its 4-momentum from p to p' ($p' = p + q$ if q is the momentum of the photon),



without loosing its character of free particle. Of course this cannot be achieved (because of energy momentum conservation) if the photon is a free photon, obeying the relation $q^2 = 0$. If, however, the photon is a virtual photon, for which the relation $q^2 = 0$ does no longer hold, the process becomes possible.

Let us consider scattering of electrons by nucleons, represented by a graph of the type



The electromagnetic field satisfies the equation:

$$\Box A_\mu = -j_\mu ,$$

where j_μ can be obtained from an approximate expression given by 1st order perturbation theory

$$j_\mu = ie\bar{u}(k)\gamma_\mu u(k) \exp[iqx] ,$$

($q = k' - k$; $\bar{u}$ and u are Dirac spinors).

Now $q^2 \neq 0$ and we find for $A_\mu(x)$

$$A_\mu(x) = ie\,\frac{\bar{u}(k')\gamma_\mu u(k)}{q^2} \exp[iqx] = A_\mu(q) \exp[iqx] . \tag{1}$$

Now our problem is just to study the interaction of a physical nucleon with a given electromagnetic field not satisfying the relation $q^2 = 0$. In this case we avoid the necessity of re-emitting the absorbed photon (as in the case $q^2 = 0$) and the consideration of complicated intermediate states, since the nucleon is always on the energy shell.

We start by studying in a little more detail the simple case of a point nucleon, with a given charge e and a given anomalous magnetic moment μ. Its wave function satisfies the Dirac equation taking account of the electromagnetic interaction:

$$\gamma_\mu \frac{\partial}{\partial x_\mu}\psi + m\psi - ie\gamma_\mu A_\mu(x)\psi - i\mu\gamma_\mu\gamma_\nu\left(\frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}\right)\psi = 0 .$$

It is convenient to go to a non-relativistic two-component equation. This was first performed by FOLDY by means of an expansion in powers of $1/m$. We get the following equation:

$$\left(m + \frac{p^2}{2m} + e\varphi + \frac{\mu}{2m}\Box\varphi - \left(\mu + \frac{e}{2m}\right)\boldsymbol{\sigma}\cdot\boldsymbol{H}\right)\psi = i\frac{\partial\psi}{\partial t} . \tag{2}$$

φ is the scalar potential, satisfying the equation

$$\Box\varphi = -4\pi\varrho , \qquad (\varrho = \text{external charge density}),$$

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H is the magnetic field. We see that in this non-relativistic approximation only σ-matrices survive, since $\gamma_4\psi \simeq \psi$. It is to be noted that the Dirac magnetic moment $e/2m$ is added to the anomalous magnetic moment μ. If $\mu \neq 0$ there is an extra term $(\mu/2m)\Box\varphi$ (the so-called Foldy term) which has to be interpreted. It describes a sort of electrostatic interaction of our particle with a given external charge distribution. We see that even a neutral particle with an anomalous magnetic moment can interact electrostatically: this is a relativistic effect (order $1/m$) very important in the study of the electron-neutron interaction, which is due not only to the neutron structure, but can occur also for point particles.

In particular we shall see that this term explains almost completely the actual electron-neutron interaction, so that the charge distribution of the neutron seems to be very concentrated, and the neutron seems to behave rather like a point particle.

Let us consider now the modifications due to the internal structure of nucleons. If we still restrict ourselves to first order perturbation theory, and look for a suitable expression of the electron scattering amplitude, we have:

$$S = \int_{-\infty}^{+\infty} \mathrm{d}t H'(t) , \qquad H'(t) = \int \mathrm{d}^3 x j_\mu(x) A_\mu(x) ,$$

and

$$\langle p' | S | p \rangle = \int \mathrm{d}^4 x j_\mu(x) A_\mu(x) = (2\pi)^4 i\delta(p + q - p')\cdot$$
$$\cdot[\overline{u}(p')(ie\gamma_\mu + i\mu\sigma_{\mu\nu}q_\nu)\, u(p)]\, A_\mu(q) ,$$

with $\sigma_{\mu\nu} = i\gamma_\mu\gamma_\nu(1 - \delta_{\mu\nu})$ in which for $A_\mu(q)$ we must put some suitable expression (as (1) for the electron-nucleon scattering).

At this point we can ask how such an expression must be modified if we want to take into account the nucleon structure.

The matrix element must be always gauge invariant and Lorentz invariant. We can think (and a detailed proof could be given) that the only way in which this expression can be altered without violating conservation laws is to consider the constants e and μ functions of q, and more precisely of the invariant q^2.

$$e = e(q^2) , \qquad \mu = \mu(q^2) .$$

Thus the conservation law $\partial j_\mu / \partial x_\mu = 0$ is still valid. These functions define what we mean by « nucleon structure » and give the charge and current distributions inside the nucleon.

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We are interested in small values of the momentum transfer q. Therefore we expand $e(q^2)$ and $\mu(q^2)$ in powers of q^2 and neglect terms in q^4. For instance we put

$$e(q^2) = \varepsilon_0 + \varepsilon_1 q^2 + \dots , \tag{3}$$

$\varepsilon_0 = e(0)$ is the total charge of the nucleon. Also the term $\varepsilon_1 q^2$ allows of a direct interpretation. If we consider in eq. (2) the terms $e\varphi + (\mu/2m)\Box\varphi$ (equal to $[e + (\mu/2m)q^2]\varphi$ for monochromatic electromagnetic fields, as described by (1)) we note that the term $\varepsilon_i q^2\varphi$ is added to the Foldy term $(\mu/2m)q^2\varphi$ (which is of the same order in q^2) and represents the contribution of the charge distribution to the electrostatic interaction of a neutral particle.

We can describe this interaction by means of a square-well potential of range equal to the classical electron radius. (Since the results depend on the product range × depth we can choose one of the two arbitrarily). In the case of the neutron it has been found that for the Foldy term the depth turns out to be 4 080 eV; for the ε_1 contribution only (80 ± 190) eV. This means that the contribution of charge distribution to the electrostatic interaction is irrelevant and that the neutron behaves as a point particle. In the first experiments electrons were scattered by thermal neutrons, *i.e.* at very low energies; in another kind of experiments electrons are scattered by deuterons (see later).

Now we are going to define the mean square radius both for charge and magnetization distribution. First we shall give an expression for these distributions. If we neglect the nucleon recoil we have $q_0 = 0$ and $q^2 = \boldsymbol{q}^2$. Now we can obtain for example the charge distribution $\varrho(r)$ simply by evaluating the Fourier 3-dimensional transform of the function $e(\boldsymbol{q}^2)$

$$\varrho(\boldsymbol{r}) = \varrho(r) = \frac{1}{(2\pi)^3}\int \exp[i\boldsymbol{q}\boldsymbol{r}]\, e(\boldsymbol{q}^2)\, \mathrm{d}^3\boldsymbol{q}.$$

Note that this function for e=constant reproduces the case of a point charge and that integrating over $\mathrm{d}^3\boldsymbol{r}$ we get the total charge $e(0)$. It is also

$$e(\boldsymbol{q}^2) = \int \mathrm{d}^3\boldsymbol{r}\, \varrho(r) \exp[-i\boldsymbol{q}\boldsymbol{r}] ,$$

and, expanding both sides in powers of $\boldsymbol{q}^2$ (note that $\int i\boldsymbol{q}\cdot\boldsymbol{r}\varrho(r)\, \mathrm{d}^3\boldsymbol{r} = 0$),

$$\varepsilon_0 + \varepsilon_1 \boldsymbol{q}^2 = \int \varrho(r)\, \mathrm{d}^3\boldsymbol{r} + \frac{1}{2}\int (-i\boldsymbol{q}\boldsymbol{r})^2\, \varrho(r)\, \mathrm{d}^3\boldsymbol{r} = \int \varrho(r)\, \mathrm{d}^3\boldsymbol{r} - \frac{1}{6} q^2 \int \varrho(r) r^2\, \mathrm{d}^3\boldsymbol{r},$$

and finally, from (3)

$$\varepsilon_1 = \left(\frac{\mathrm{d}e(q^2)}{\mathrm{d}q^2}\right)_{q^2=0} = -\frac{1}{6}\int \varrho(r)\mu^2\, \mathrm{d}^3\boldsymbol{r} = -\frac{1}{6}\overline{r_1^2} ,$$

where $\overline{r_1^2}$ is the mean square radius for the charge distribution.

Analogously we find

$$\mu_1 = \left(\frac{\mathrm{d}\mu(q^2)}{\mathrm{d}q^2}\right)_{q^2=0} = -\frac{1}{6}\overline{r_2^2}\,.$$

The indices 1 and 2 are used because generally one adopts the notation $e(q^2) = F_1(q^2)$ and $\mu(q^2) = F_2(q^2)$ which will be used in the following. Thus we can obtain information about the first derivatives of the F's if we know the mean square radii of the electric and magnetic distributions, which can be determined experimentally.

From Hofstadter experiments on protons and deuterons we get (the index p and n stand for proton and neutron)

$$r^2_{1n} \simeq (0.003 \pm 0.007)\cdot 10^{-26}\ \mathrm{cm}^2\,,$$

$$F_{1p} \simeq F_{2p}\,, \qquad \text{for } q^2 \lesssim 9\cdot 10^{-26}\ \mathrm{cm}^{-2},$$

and

$$\overline{r^2_{1p}} \simeq \overline{r^2_{2p}} = \frac{1}{\mu^2}(0.32 \pm 0.03)\,.$$

Here μ = pion mass $= 0.7\cdot 10^{13}\ \mathrm{cm}^{-1}$. If we compare this data, we see that the value of $\overline{r^2_{1n}}$ is practically 0 experimentally, and in any case is negligible: that is, as previously stressed, the neutron shows a point-particle behaviour. The values for the proton have not been measured directly: a suitable analytic expression for F_1 and F_2 has been postulated with a parameter to fit the experimental curves up to high values of momentum transfer, where the expansion in powers of q^2 is no longer correct. Thus some doubt could be cast on the reliability of these results: one might think that, even though the expression of these functions is the best possible for the given momentum transfer interval, it could fail for the lowest values of q^2, in which we are interested. This refers particularly to the magnetic moment, whose effects become sensitive only at higher momentum transfers.

As far as F_{1n} and F_{2n} are concerned, the results are consistent with $F_{1n} = 0$ even for higher momentum transfer. An upper limit is $F_{1n} \leqslant 0.3 F_{1p}$.

F_{2n} seems to be comparable with F_{1p} and F_{2p}. A lower limit is $F_{2n} \geqslant 0.7 F_{2p}$.

The fact that the neutron and proton charge distributions are so different brings some difficulty from the theoretical point of view, since it is not easily seen how in the case of the neutron the charge effect of the meson cloud (which we know to be different from zero from the proton results) could be cancelled by the charge effect of the core. SALZMAN has carried out some calculations on the basis of the fixed-source model and has found that the neutron should have a core whose radius is almost as large as the radius of the meson cloud. In order to understand the problem in a more satisfactory way, we can use dispersion relations. Rewriting the form factors F_1 and F_2 in terms of isotopic spin we put: $F_{1,2}=F^S_{1,2}+\tau_3 F^V_{1,2}$ and clearly we get $F_{1p}=F^S_1+F^V_1$; $F_{1n}=F^S_1-F^V_1$ (and a similar formula for F_2). The indices S and V indicate that $F^S_{1,2}$ and $F^V_{1,2}$ behave respectively like a scalar and a vector in the isotopic spin space. We have also

$$\overline{r^2_{1p,n}}=\tfrac{1}{2}[(r^S_1)^2\pm(r^V_1)^2]\,, \qquad \text{proton} + \text{sign; neutron} - \text{sign},$$

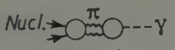
and a similar formula for $\overline{r^2_2}$.

Thus, if $(r^S_1)^2\simeq\overline{(r^V_1)^2}$ it turns out that $\overline{r^2_{1n}}\simeq 0$, while $\overline{r^2_{1p}}$ is large. The dispersion relations are (from CHEW *et al.*, and GOLDBERGER *et al.*):

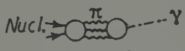
$$F^S_1(q^2)=\frac{e}{2}-\frac{q^2}{\pi}\int\limits_{(3\mu)^2}^{\infty}\mathrm{d}m^2\,\frac{\varrho^S_1(m^2)}{m^2(m^2+q^2)}\,, \qquad F^V_1(q^2)=\frac{e}{2}-\frac{q^2}{\pi}\int\limits_{(2\mu)^2}^{\infty}\mathrm{d}m^2\,\frac{\varrho^V_1(m^2)}{m^2(m^2+q^2)}\,,$$

$$F^S_2(q^2)=\frac{1}{\pi}\int\limits_{(3\mu)^2}^{\infty}\mathrm{d}m^2\,\frac{\varrho^S_2(m^2)}{m^2+q^2}\,, \qquad F^V_2(q^2)=\frac{1}{\pi}\int\limits_{(2\mu)^2}^{\infty}\mathrm{d}m^2\,\frac{\varrho^V_2(m^2)}{m^2+q^2}\,.$$

The functions ϱ are the imaginary parts of the F's and are still unknown. However one can attempt to calculate them, taking into account only the simplest intermediate states which can be put in the vertex part of the photon-nucleon interaction. We see that the integrals of the scalar part have $(3\mu)^2$ as the lower limit, while the integrals of the vector part have $(2\mu)^2$. This follows from the fact that, as can be proved, the intermediate states of the scalar part cannot contain less than three mesons and the intermediate state of the vector part cannot contain less than two mesons. Intermediate states with one meson do not occur. The simplest graphs which can be used to calculate the functions ϱ are therefore the following:



V part



S part

Calculations for the scalar contribution have not yet been carried out. For the vector contribution the situation is simpler. The left hand part of the diagram can be in some way related to the matrix element for pion-nucleon scattering. The right-hand part implies a knowledge of the electromagnetic structure of the pion, which is very difficult to detect experimentally. CHEW *et al.*, and GOLDBERGER *et al.*, have carried out calculations which give good agreement for the vector anomalous magnetic moment, but give far too

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small a value for the mean square radius of the charge distribution (by a factor $\frac{1}{2} \sim \frac{1}{3}$).

It may be that the only way of explaining the large value of the mean square radius of the proton charge distribution is to admit that the electromagnetic structure of the pion does not extend only to distances of the order of the nucleon's Compton wavelength but to larger distances. This would induce a strong pion-pion interaction, which could manifest itself in other types of phenomena. In fact it seems that there are some indications of such an interaction in phenomena such as τ-decay or pion production in nucleon-antinucleon annihilation. An other difficulty to solve is that the isotopic scalar part of the F's must be large in the case of the charge distribution and small in the case of the magnetization distribution, in order that the experimental results may be explained.

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Symmetry Properties of Elementary Particles.

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The problem one is faced with in the study of elementary particles resembles in a way that met in solid state physics: one has to solve a many body problem, which in itself is very difficult. However, a number of statements can be made, using only symmetry properties which restrict the dynamics at hand.

We shall deal with quantum field theory which forms the basis of the present concepts about elementary particles.

Outline of quantum field theory from fundamental principles.

Current field theories are examples of the application of quantum mechanics to systems covariant under the Lorentz group. It happens that relativistic field theories exhibit unexpected features (*e.g.* TCP theorem, connection between spin and statistics) which have to be considered as fundamental laws of nature. It is certainly of interest to investigate such « symmetry » properties.

1. – Invariance under the connected homogeneous Lorentz group.

Let us consider field quantities $\varphi_\alpha(x)$ which behave in a prescribed way under Lorentz transformations, *i.e.*, a Lorentz transformation $x \to x' = Ax$ $(x'_i = A_{ik}x_k)$, induces a linear transformation

$$\varphi_\alpha(x) \to \varphi'_\alpha(x') = l_{\alpha\beta}\varphi_\beta(x)\,.$$

(*) The following are notes taken by Drs. R. STORA and B. UDGAONKAR. The author is indebted to them for carrying out this unrewarding task.

One considers only linear representations, since only these have so far found application in physics. It is clear that the knowledge of all such representations would yield all the possible relativistically covariant systems which can be constructed; one needs consider only irreducible representations. We shall not discuss the general problem of the construction of such representations, but only consider some of the simplest which appear to be of practical use.

2. – Scalar field.

One obtains a trivial representation of the Lorentz group by considering a scalar field $\varphi(x)$ which transforms according to:

$$\varphi(x) \to \varphi'(x') = \varphi(x) . \tag{2.1}$$

It is useful to concentrate on the Lagrangian. We have to find a scalar which yields covariant equations of motion through the variational principle:

$$L = \tfrac{1}{2}(\partial_\mu\varphi\,\partial^\mu\varphi - m^2\varphi^2) . \tag{2.2}$$

This is the most general Lagrangian bilinear in the field and containing only first order derivatives. It implies the free field equation of motion:

$$(\partial_\mu\partial^\mu + m^2)\varphi = 0 . \tag{2.3}$$

3. – The unimodular representation.

Consider two by two unimodular matrices M such that $|M| = \det M = 1$. These matrices clearly depend on six parameters, and we shall show that they give a double-valued representation of the Lorentz group. It can be constructed with the aid of the four basic Pauli matrices:

$$\sigma_\mu: \qquad \sigma_0 = 1,\ \sigma_1,\ \sigma_2,\ \sigma_3 . \tag{3.1}$$

Given a vector d_μ, consider the matrix $\sigma_\mu d^\mu$:

$$|\sigma_\mu d^\mu| = d^\mu d_\mu = {d^0}^2 - {d^1}^2 - {d^2}^2 \dots . \tag{3.2}$$

Then one sees that:

$$M^+\sigma^\mu d_\mu M = \sigma^\mu d'_\mu , \tag{3.3}$$

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where (d'_μ) is a vector which can be obtained from (d_μ) by a Lorentz transformation since:

$$d'^\mu d'_\mu = |\sigma^\mu d'_\mu| = |M^+ \sigma^\mu d_\mu M| = d^\mu d_\mu . \tag{3.4}$$

We shall see later that conversely, to each Lorentz transformation, there correspond two unimodular transformations. From this representation, one can construct Lorentz covariant theories. In fact, if one has a two component complex spinor field $\psi(x)$ transforming according to:

$$\psi(x) \rightarrow \psi'(x') = M\psi , \tag{3.5}$$

then we know that $\psi^+ \sigma^\mu \psi$ transforms like a vector and one can verify that $\psi_\lambda g_{\lambda\beta} \psi_\beta$ and $\psi^+_\lambda g_{\lambda\beta} \psi^+_\beta$ are invariant, where $g = \sigma_2$ (use the relation $M^T \sigma_2 M = \sigma_2$). One can construct the hermitian Lagrangian for free fields:

$$L = \tfrac{1}{2}\left[i(\psi^+ \sigma_\mu \partial^\mu \psi - (\partial_\mu \psi^+)\sigma^\mu \psi) + \psi g \psi + \psi^+ g \psi^+\right] , \tag{3.6}$$

(we put the mass equal to one for simplicity). The corresponding field equations are:

$$i\sigma_\mu \partial_\mu \psi + g\psi^+ = 0 , \qquad (\text{also } \partial_\mu \partial^\mu \psi + \psi = 0) . \tag{3.7}$$

We shall see that this theory is equivalent to the so called Majorana theory.

For our purpose this presentation is preferable to the usual one, since the well known argument of Dirac about the linearization of the Klein-Gordon equation looses much of its force in view of the difficulties occurring in the one-particle theory. The present approach has the advantage that it is based on a systematic search for a relativistic theory.

4. – Quantization.

The free fields we have constructed can be regarded as systems of infinitely many oscillators coupled to their neighbours. As these oscillators can be analyzed into simple harmonic decoupled ones, it is valuable to reduce the problem to one degree of freedom. We shall follow the approach taken by BURTON and TOUSCHEK, and extensively used by SCHWINGER; this has the advantage of treating both Bose-Einstein and Fermi-Dirac systems on the same footing. Consider the simple harmonic motion:

$$\ddot{q} = -q . \tag{4.1}$$

One can equally well use the variable:

$$\tilde{Q} = q + i\dot{q} . \tag{4.2}$$

Then:

$$\dot{\tilde{Q}} = i\tilde{Q} . \tag{4.3}$$

It is preferable to use the real components:

$$Q = \begin{pmatrix} q \\ \dot{q} \end{pmatrix} \qquad \dot{Q} = aQ \qquad \text{where} \qquad a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{1.4.4}$$

(a represents a real realization of $\sqrt{-1}$).

In order to construct a Hamiltonian, we look for a bilinear form $H = \frac{1}{2}QBQ$ such that $\dot{H} = 0$. This implies: $[B, a] = 0$, so that, either $B = 1$ or $B = a$. We thus get the two cases:

$$\begin{cases} \text{Bose-Einstein:} & H = \frac{1}{2}Q1Q . \\ \text{Fermi-Dirac:} & H_F = \frac{1}{2}iQaQ . \end{cases} \tag{4.5}$$

(In the classical theory, $H_F = \frac{1}{2}i(-\dot{q}q + q\dot{q}) = 0$, since q and $\dot{q}$ commute). The quantization goes through by requiring that:

$$[H, Q] = -i\dot{Q} . \tag{4.6}$$

One has:

$$\begin{aligned} \tfrac{1}{2}[Q_\alpha B_{\alpha\beta} Q_\beta, Q]_- &= \tfrac{1}{2}\{Q_\alpha [B_{\alpha\beta} Q_\beta, Q]_\mp - [Q, Q_\alpha]_\mp B_{\alpha\beta} Q_\beta\} , \\ &= -iaQ . \end{aligned} \tag{4.7}$$

The simplest solution to this equation is obtained by assuming that:

$$[Q_\alpha, Q_\beta]_\mp = C_{\alpha\beta} = c\text{-number} , \tag{4.8}$$

C either symmetric $(+)$ or antisymmetric $(-)$.

Then one finds:

$$\tfrac{1}{2}\{(BC)^T - CB\} = -ia ,$$

Bose-Einstein case:

$$B = 1, \qquad \tfrac{1}{2}(C^T - C) = -ia \rightarrow C = ia ,$$

thus C is antisymmetric, so that one has to use the commutator in (4.8):

$$[q, \dot{q}] = i\,.$$

Fermi-Dirac case:

$$B = ia\,, \qquad \tfrac{1}{2}(C^T + C) = 1 \rightarrow C = 1\,,$$

thus C is symmetric, and one has the anticommutation rules:

$$[q, q]_+ = [\dot{q}, \dot{q}]_+ = 1 \qquad [q, \dot{q}]_+ = 0\,. \tag{4.9}$$

Thus we find that the two simplest cases of harmonic motion correspond to the following systems:

Bose-Einstein: « Classical oscillator »:

$$H = \tfrac{1}{2}[q^2 + \dot{q}^2]\,, \qquad [q, \dot{q}] = i\,.$$

Fermi-Dirac: « Larmor precession »:

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix}\sigma_x \\ \sigma_y\end{pmatrix} \rightarrow \sigma_x^2 = \sigma_y^2 = 1\,, \quad [\sigma_x, \sigma_y]_+ = 0\,, \quad H = \sigma_z = i\sigma_x\sigma_y\,,$$

which describes the precession of a spin in a magnetic field. The Lagrangians corresponding to the above Hamiltonians are of the form:

$$L = \tfrac{1}{2}(QA\dot{Q} - \dot{Q}AQ) - QBQ\,,$$

where:

Bose-Einstein: $\begin{cases} B = \tfrac{1}{2} & \text{(real symmetric)}\,, \\ A = -\tfrac{1}{2} & \text{(real antisymmetric)}\,, \end{cases}$

Fermi-Dirac: $\begin{cases} B = \tfrac{1}{2}i & \text{(imaginary antisymmetric)}\,, \\ A = \tfrac{1}{2}i & \text{(imaginary symmetric)}\,, \end{cases}$

and

$$[Q_\alpha, Q_\beta]_\mp = \tfrac{1}{2}i(A^{-1})_{\alpha\beta}\,.$$

We shall see later that the symmetry of these matrices implies the connection between spin and statistics.

5. – Generalization to fields.

Instead of Q, we introduce the multicomponent field $\chi(x)$ and we replace $\dot{Q}$ by the four space-time derivatives $\partial_\mu \chi(x)$; we then have the Lagrangian:

$$L = \tfrac{1}{2}(\chi A^\mu \partial_\mu \chi - \partial_\mu \chi A^\mu \chi) - \chi B \chi \,, \tag{5.1}$$

where the five matrices A^μ and B are real antisymmetric, or, imaginary symmetric and real symmetric or imaginary antisymmetric.

The commutation relations are:

$$[A^0_{\alpha\gamma}\chi_\gamma(x),\, \chi_\beta(x')]_{x^0=x^{0'}} = \frac{i}{2}\delta(x-x')\delta_{\alpha\beta}\,, \tag{5.2}$$

or

$$[\chi_\alpha(x),\, \chi_\beta(x')]_{x^0=x^{0'}} = i(A^{(0)-1})_{\alpha\beta}\delta(\chi-\chi')\,,$$

if A^0 is non singular. As $\chi(x)$ replaces the former hermitian Q, it must itself be hermitian.

6. – Applications.

6·1. *Klein-Gordon equation.* – One introduces $\chi(x) = \begin{pmatrix} \varphi(x) \\ \varphi_\mu = \partial_\mu \varphi \end{pmatrix}$ and obtains the first order form:

$$A_\mu \partial_\mu \chi(x) = B\chi(x)\,, \tag{6.1}$$

with:

$$B = 1\,, \qquad A^0 = \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}.$$

that is to say:

$$-\partial_\mu \varphi^\mu = \varphi\,, \qquad \varphi_\mu = \partial_\mu \varphi\,. \tag{6.2}$$

The disadvantage of this formulation (Duffin-Kemmer) is that $A^{(0)}$ is singular, so that not all the variables are kinematically independent. (One has both equations of motion and constraints).

6·2. *Majorana field.* – The ψ-field used in Eq. (3.5) corresponds to the non hermitian operator $\tilde{Q} = q + i\dot{q}$; one can go to the hermitian description (similar to Q) by introducing:

$$\chi(x) = \begin{pmatrix} \mathrm{Re}\ \psi_1(x) \\ \mathrm{Im}\ \psi_1(x) \\ \mathrm{Re}\ \psi_2(x) \\ \mathrm{Im}\ \psi_2(x) \end{pmatrix} \qquad \text{where} \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{6.3}$$

The realization of $\sqrt{-1}$ used here is the antisymmetric matrix $-i\varrho_2 = \gamma_5$.

The Lagrangian can be rewritten in the form (5.1) with:

$$A^{(0)} = i\,, \qquad A^{(1)} = i\sigma_1\,, \qquad A^{(2)} = i\sigma_2\varrho_2\,, \qquad A^{(3)} = i\sigma_3\,, \qquad B = \varrho_3\sigma_2\,,$$

the A's are pure imaginary and symmetric; B is pure imaginary and antisymmetric.

Thus, one has to use the Fermi-Dirac quantization.

We clearly see the equivalence between the Weyl description in terms of the complex two-component field ψ, and the Majorana description in terms of the real four-component field χ, which are respectively analogous to the description of the elementary oscillator in terms of the non hermitian variable Q and the hermitian Q.

7. – Connection between spin and statistics.

The foregoing analysis has shown that the free irreducible Lagrangian constructed for spin 0 and $\frac{1}{2}$ necessitates the use of A-matrices with well defined symmetry properties. These symmetry properties imply that commutation relations pertain to the prior case, whereas anticommutation rules pertain to the second one; otherwise, the Lagrangians constructed would reduce to c-numbers. Thus, under the assumption that only commutators and anticommutators are to be used, we have arrived, by the very construction of these theories, to a connection between spin and statistics which does not make use of such physical requirements as the positive definiteness of energy.

Whereas it is presumed that this connection is fundamental, one can say that the way it comes into the formalism cannot be related to any intuitive picture.

8. – Remark on the structure of commutation rules.

Although the commutation rules for scalar and spinor fields differ by the appearance of derivatives in the latter case, they are quite close in structure;

the commutation rules for spin 0 fields are

$$[\varphi(x), \varphi(x')]_- = i\Delta(x-x') . \tag{8.1}$$

One cannot use anticommutation rules: one would have one of the following:

$$\begin{cases} [\varphi(x), \varphi(x')]_+ = \dot{\Delta}(x-x') , \\ [\varphi(x), \varphi(x')]_+ = \Delta'(x-x') . \end{cases} \tag{8.2}$$

The first follows from application of (4,9) but is not Lorentz covariant, the second is not casual.

For spin $\frac{1}{2}$ fields, one has:

$$[\psi_\alpha(x), \psi_\beta(x')]_{+\,x_0=x'_0} = \tfrac{1}{2}\delta_{\alpha\beta}\delta(x-x') , \tag{8.3}$$

in the hermitian representation, while in the 2-component formulation, the rules are:

$$\begin{cases} [\psi_\alpha(x), \psi_\beta(x')]_+ = g_{\alpha\beta}\,\Delta(x-x') , \\ [\psi^+_\alpha(x), \psi_\beta(x')]_+ = i\,\sigma^\mu_{\alpha\beta}\partial_\mu\,\Delta(x-x') , \end{cases} \tag{8.4}$$

where it is clear that, in the second line, $\psi^+ = -ig\sigma^\mu\partial_\mu\psi$ plays the rôle of $\dot{\varphi}$ and the latter commutation rules are quite analogous to:

$$[\varphi(x), \dot{\varphi}(x')]_- = \dot{\Delta}(x-x') .$$

Thus, one may say in a loose way that if ψ (or ψ^+) is considered as the basic field, ψ^+ (or ψ) is essentially its derivative. That ψ and ψ^+ must be put on the same footing is clear as both must be symmetrically involved in the hermitian Lagrangian.

This remark may be of importance when, in writing weak coupling terms, one may say that the interactions containing two ψ's and two ψ^+'s involve the minimum number of derivatives, whereas a term involving four ψ's must be supplemented by one involving four ψ^+'s, in order to preserve hermiticity, that is to say involve a higher number of derivatives.

The presence of derivatives, however, prevents the theory from being renormalizable.

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9. – Parity.

We have to come back to some peculiarities exhibited by the unimodular representation: it is well known that the matrices representing rotations and Lorentz transformations are:

$$(9.1)\qquad \begin{cases} L_{\text{rotation}} = \exp\left[(\varphi/2)\,\boldsymbol{\alpha}\cdot\boldsymbol{n}\varrho_2\right], & \text{for a rotation } (\boldsymbol{n}, \varphi), \\ L_{\text{Lorentz}} = \exp\left[(\varphi/2)\,\boldsymbol{\alpha}\cdot\boldsymbol{n}i\right], & \text{for Lorentz trans. } (n, v = \operatorname{tgh}\varphi). \end{cases}$$

These matrices are real, but do not belong to the orthogonal group. One can instead say that they are orthogonal whith respect to the indefinite matrix β according to:

$$(9.2)\qquad L^* = L\,, \qquad L^T\beta L = \beta\,.$$

The reality implies that, if one forms a Dirac field out of two Majorana fields: $\psi = \psi_1 + i\psi_2$, the two Majorana fields transform separately, *i.e.* the Dirac field does not provide an irreducible representation.

On the other hand, we see that, by performing a 360° rotation one changes the sign of a spinor since:

$$(9.3)\qquad L_{\text{rot}}(360^\circ) = -1\,.$$

As well as rotations, space reflections leave the free field theory covariant, namely the replacement

$$(9.4)\qquad \psi(\boldsymbol{x}, x^0) \to \psi'(-\boldsymbol{x}, x^0) = i\beta\psi(-\boldsymbol{x}, x^0)\,,$$

which is to say both the equations of motion and the commutation rules are left invariant.

Therefore, there exists a unitary operator R such that

$$(9.5)\qquad R^{-1}\psi(\boldsymbol{x}, x^0)R = i\beta\psi(-\boldsymbol{x}, x^0)\,.$$

From the properties of β: $\beta^2 = 1$, $\beta^T = -\beta$, one gets:

$$(9.6)\qquad R^{-2}\psi(\boldsymbol{x}, x^0)R^2 = -\psi(\boldsymbol{x}, x^0)\,,$$

which is to say that two reflections are equivalent to a 360° rotation, which is reasonable. One then can say that a spinor field has intrinsic parity i ($(i\beta)^2 = -1$). Although selection rules (angular momentum, charge conservation) prevent the measurement of parity for one spin $\frac{1}{2}$ particle, the intrinsic parity of two spin $\frac{1}{2}$ particles, if measurable, is -1. (*e.g.* e^+e^- have intrinsic, parity -1, which implies that in the decay of the S ground state of positronium, the two produced γ-rays must have orthogonal polarizations; this has been verified experimentally; unfortunately, this is more difficult to do for π^0 decay).

The parity assignments in the case of reducible systems involve more arbitrariness since, if Q is a constant of the motion, P or for instance $P(-)^Q$ are just as good in order to define a parity operation.

10. – C.P.T. theorem.

When couplings are introduced, it may very well be that the invariance under space reflections is violated. However, there is a reflection invariance which holds for theories invariant under the proper orthochronous Lorentz group: introduce the space-time reflection operator K such that:

$$(10.1)\qquad \begin{cases} (a) & KiK^{-1} \;\;\; = -i\,, \qquad\qquad (K \text{ is antiunitary}), \\ (b) & K\varphi(x)K^{-1} = \varphi(-x)\,, \\ (c) & K\psi(x)K^{-1} = i\gamma_5\psi(-x)\,. \end{cases}$$

Then we find: $KT^{(n)}(x)K^{-1} = (-)^n T^{(n)}(-x)$ where $T^{(n)}(x)$ is a tensor of order n; therefore, for any Lorentz invariant interaction: $KL'K^{-1} = L'$. Then, from the definitions of energy-momentum, angular momentum, charge, one finds:

$$(1.10.2)\qquad \begin{cases} (a) & KP_kK^{-1} \;\; = K\displaystyle\int \mathrm{d}V T_{k0}K^{-1} = P_k\,, \\ (b) & KM_{ij}K^{-1} = K\displaystyle\int \mathrm{d}V M_{ij0}K^{-1} = -M_{ij}\,, \\ (c) & KQK^{-1} \;\;\; = K\displaystyle\int \mathrm{d}V J_0 K^{-1} = -Q\,. \end{cases}$$

Now, if one wants to claim that the underlying symmetries of a Lorentz invariant theory are those of the corresponding theory written in terms of an euclidian continuum, one sees that it must admit K as a symmetry operation, since K is a « rotation ». However, such an extrapolation to an euclidian space requires some analytic extension, since it necessitates the replacement $x^0 \to ix^0$, which is by no means intuitive.

Example. – Consider massless spin $\frac{1}{2}$ particles; then, the solution of the equation $\sigma^\mu \partial_\mu \psi = 0$ shows that two possible polarization states can be found, namely,

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for a particle of given momentum, states with spin along or antiparallel to the momentum. So, we see a manifestation of the T.C.P. theorem: whereas one can very well imagine that there is a covariant theory with only one polarization state, we see that this one exhibits also the opposite polarization state; if PT is a symmetry operation, then the two states may have the same charge, although in general they will have opposite charges.

11. – Internal symmetry properties.

We have so far dealt with invariance properties under the Lorentz group, this is to say with space time degrees of freedom. More many symmetry properties are however exibited by known systems (*e.g.* charge conservation; isotopic spin conservation). Such properties connected with internal degrees of freedom can be accounted for in the following way: consider N identical fields with mass m; the free one-particle Lagrangian is:

$$L_i = \tfrac{1}{2}(\chi^{(i)}_{,k} A^k \chi^{(i)} - \chi^{(i)} A_k \chi^{(i)}_{,k}) + m_i \chi^{(i)} B \chi^{(i)} = \chi^{(i)} D_i \chi^{(i)} . \tag{11.1}$$

Thus the Lagrangian of the system is:

$$L = \sum_i \chi^{(i)} D_i \chi^{(i)} ; \tag{11.2}$$

if all masses are equal, all D_i are equal.

The commutation rules are:

$$[\chi_i, \chi_j]_\pm = \delta_{ij} \cdot 1 \text{ (space)} . \tag{11.3}$$

In the cases of equal masses, the Lagrangian is invariant under the substitution

$$\chi_i \to \chi'_i = \Lambda_{ij}\chi_j \qquad \text{where} \qquad \Lambda^T \Lambda = 1 . \tag{11.4}$$

The commutation rules are invariant under the same transformation provided (11.5) $\Lambda\Lambda^T = 1$; the hermiticity of the fields is preserved

$$\Lambda^* = \Lambda . \tag{11.6}$$

Then there exists a unitary operator $U(\Lambda)$ such that

$$U^{-1}(\Lambda)\chi U(\Lambda) = \Lambda\chi , \tag{11.7}$$

and $U(\Lambda)$ is a constant of the motion since, from the bilinearity of the energy momentum:

$$U^{-1}(\Lambda)P_kU(\Lambda)=P_k \quad \text{and also} \quad U^{-1}(\Lambda)J_{ik}U(\Lambda)=J_{ik}\,. \tag{11.8}$$

Infinitesimal orthogonal transformations can be written as:

$$\Lambda_{ij}=\delta_{ij}+\sum_r \delta_\alpha^{(r)}T_{ij}^{(r)}\,, \tag{11.9}$$

where the T matrix is antisymmetrical, real; there are therefore $N(N-1)/2$ such indipendent matrices which give rise to $N(N-1)/2$ constants of the motion, since the infinitesimal version of $U(\Lambda)$ can be written:

$$U(\delta\Lambda)=1+i\sum_r \delta\alpha^{(r)}\int \mathrm{d}\sigma^\mu \chi^{(i)}A_\mu T_{ij}^{(r)}\chi^{(j)}\,. \tag{11.10}$$

This follows from Schwinger's variational principle and can also be verified by direct calculation.

One can then define currents

$$J_\mu^r=\chi_iA_\mu T_{ij}^{(r)}\chi_j\,, \tag{11.11}$$

which obey a conservation law. Since

$$\left[\int J_\mu^{(r)}\mathrm{d}\sigma^\mu,\,P^0\right]=\frac{\mathrm{d}}{\mathrm{d}t}\int_{t\,=\,\text{const.}}\mathrm{d}\sigma^\mu J_\mu^{(r)}=0\,,$$

implies

$$\partial^\mu J_\mu^{(r)}=0\,. \tag{11.12}$$

Depending on the coupling, the theory may be invariant under some discrete transformations which cannot be obtained by continuing the above continous transform,

Thus in general, theories are invariant under some subgroups of the orthogonal group, supplemented by reflection operations.

12. – Examples.

— *Given 3 «vector» fields*, a coupling $L'=\boldsymbol{\psi}\times\boldsymbol{\psi}\cdot\boldsymbol{\varphi}$ is invariant under rotations, but not under reflections.

— *Given 2 fields φ_1 and φ_2*, $L'=\varphi_1^4+\varphi_2^4$ is only invariant under the discrete operations

$$\varphi_1\to\pm\varphi_1\,,\qquad \varphi_2\to\pm\varphi_2\,,\qquad \varphi_1\leftrightarrow\pm\varphi_2\,;$$

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By the same general argument a reflection invariance implies also the existence of a constant of the motion U.

— *Two internal degrees of freedom*: electric charge.

Introduce Pauli matrices in internal space. Then one has invariance under the substitutions:

$$\left\{\begin{array}{ll} \chi \to \exp[i\omega\tau_2]\chi & \text{gauge group}\,, \\ \chi \to \tau_3\chi & \text{charge conjugation}\,. \end{array}\right. \tag{12.1}$$

The current associated with the ω-gauge group is:

$$J^{(\omega)}_{\mu} = \chi A^k \tau_2 \chi\,, \tag{12.2}$$

and the corresponding invariant interaction Hamiltonian with the electromagnetic vector potential φ^k

$$L' = \varphi^k(x)\chi A^k \tau_2 \chi\,,$$

is still invariant under this gauge group, as well as under charge coniugation provided one supplements the latter by $A_k(x) \to -A_k(x)$.

— 4 *Fermi fields interacting with three Bose fields* (pion-nucleon system).

In order to represent the 6 infinitesimal operations corresponding to 4 Fermions by antisymmetric matrices one needs at least 4×4 matrices, out of which one can select six antisymmetric matrices $\boldsymbol{r}$, $\boldsymbol{s}$. Introducing two sets $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ of Pauli matrices, one can write:

$$\left\{\begin{array}{ll} \boldsymbol{r} = i\mu_2(\nu_1\,,\ \nu_2\mu_2\,,\ \nu_3)\,, & \text{real antisymmetric}, \\ \boldsymbol{s} = i\nu_2(\mu_1\,,\ \mu_2\nu_2\,,\ \mu_3)\,, & ([r_i\,, s_j] = 0\,, \quad [r_i\,, r_j] = \varepsilon_{ijk}r_k)\,. \end{array}\right. \tag{12.4}$$

To describe the coupling of nucleons with π-mesons, one must write:

$$L' = \underset{\substack{\text{meson}\\ \text{field}}}{\boldsymbol{\varphi}} \cdot \underset{\substack{\text{Ps. scal.}\\ \text{coupling}\\ \text{A. S.}}}{\chi\varrho_2\ i\beta} \ \underset{\text{A. S.}}{\boldsymbol{s}} \ \underset{\text{A. S.}}{r_2}\chi\,. \tag{12.5}$$

The general orthogonal transformation for 4 fermions is: $\chi \to \exp[\boldsymbol{n}_1\boldsymbol{s}+\boldsymbol{n}_2\boldsymbol{r}]\chi$. This coupling is invariant under:

$$\left\{\begin{array}{lll} \left.\begin{array}{ll} (a) & \chi \to \exp[\boldsymbol{n}_1\boldsymbol{s}]\chi \\ (b) & \varphi \to \Lambda\varphi \end{array}\right\} & & \begin{array}{l} \text{corresponds to isotopic spin con-} \\ \text{servation (infinitesimal generator } \boldsymbol{T}\text{)}, \end{array} \\ (c)\quad \chi \to \exp[\omega r_2]\chi & & \text{baryon number conservation}, \\ (d)\ \left.\begin{array}{l} \chi \to r_2\chi \\ \varphi \to -\varphi \end{array}\right\} & R & \left.\begin{array}{l} \\ \\ \end{array}\right\} \begin{array}{l} \text{(charge conjugation reflection} \\ \text{around isotopic axis 2):} \end{array} \end{array}\right. \tag{12.6}$$

— *Case of massless field.*

In the case of a vanishing mass spin ½ field, one has additional symmetries:

$$L = \sum_i \psi_i^+ \sigma^\mu \partial_\mu \psi_i \, . \tag{12.7}$$

This Lagrangian is invariant under unitary rather than ortogonal transformations. The infinitesimal transformations are N^2 antihermitian matrices.

For $N = 2$ one obtains 4 infinitesimal generators; the most general unitary trasformation can be written:

$$\psi \to \exp\left[i(\alpha_\nu + \boldsymbol{\alpha}\cdot\boldsymbol{\tau})\right]\psi \, , \tag{12.8}$$

where the α_ν are the four real parameters of the group. The corresponding « currents » N, $\boldsymbol{T}$ have the same commutation properties as the matrices 1 and τ.

However there is no internal operation, which corresponds to charge conjugation, which has the property $[R, N]_+ = [R, T]_- = 0$.

— *Representation of the Algebra of internal symmetry properties.*

We would like here to mention the following important problem: given some internal symmetry properties, what is the minimum number of fields needed in order to describe them? In particular, can one introduce less many fields than is usually done?

a) Baryon number, isotopic spin charge conjugation.

One wants to find representations to the following algebra:

$$\begin{cases} [N, \boldsymbol{T}] = 0 \, , \\ \boldsymbol{T} \times \boldsymbol{T} = i\boldsymbol{T} \, , \\ [R, N]_+ = 0 = [R, T] \, , \end{cases} \tag{12.9}$$

it is found that one needs at least 4 fields.

b) Baryon number, isotopic spin, R, strangeness; these represent all constants of the motion for strong interactions which are known so far.

One has to supplement the previous algebra with U such that:

$$[U, N] = 0 \, , \qquad [U, T] = 0 \, , \qquad [U, R]_+ = 0 \, . \qquad [12.10]$$

Then one finds that 6 real fields are needed, or 3 complex fields. If one does not require invariance under R, only 3 mass zero fields are sufficient.

SUPPLEMENTO AL VOLUME XIV, SERIE X
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4° Trimestre

Research on High Energy Negative π-Mesons in Nuclear Emulsion (*).

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1. – High energy negative pion beam.

π-meson research can be carried out at various energies. Near zero energy measurements have been performed on π-mesic-atoms. At higher energies up to a few hundred MeV the scattering amplitudes have been determined and at still higher energies one can look at the various production processes *e.g.* those for pions, K-mesons and hyperons.

In the following a few considerations will be presented about very high-energy π-mesons which may give rise to some interesting production processes. The discussion is related to a stack of emulsion which was exposed to a high energy negative π-meson beam at the Bevatron in Berkeley.

In order to calculate the highest energy of the π-mesons produced in the target by the internal proton beam one considers the reaction:

$$p+n \to p+p+\pi^- ,$$

The incident proton having a kinetic energy of 6.2 GeV hits a neutron in a nucleus of the target. One has to take into account the motion of the neutron

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in the nucleus. When the neutron has the maximum momentum (taken to be 210 MeV/c) opposite to the incident proton, the energy in the c.m.s. reaches a maximum.

The highest energy of the outgoing pion is obtained when the two outgoing protons move in the c.m.s. with equal and parallel velocity opposite to the incident proton. The pion then moves along the direction of the incident proton and one calculates for the energy of the pion in the laboratory system 5.9 GeV. At this energy the expected intensity, on account of considerations of phase space, is too low for experiments, so that one is confined to a somewhat lower energy.

The beam used for the exposure was extracted from the Bevatron and focussed by means of two quadrupole lenses. It was deflected by a magnet and stripped by two collimators. The energy of the beam when arriving at the stack (called AF stack) varied linearly between 5.0 and 5.8 GeV. This was computed from the data of the magnetic deflection systems and the known Bevatron field.

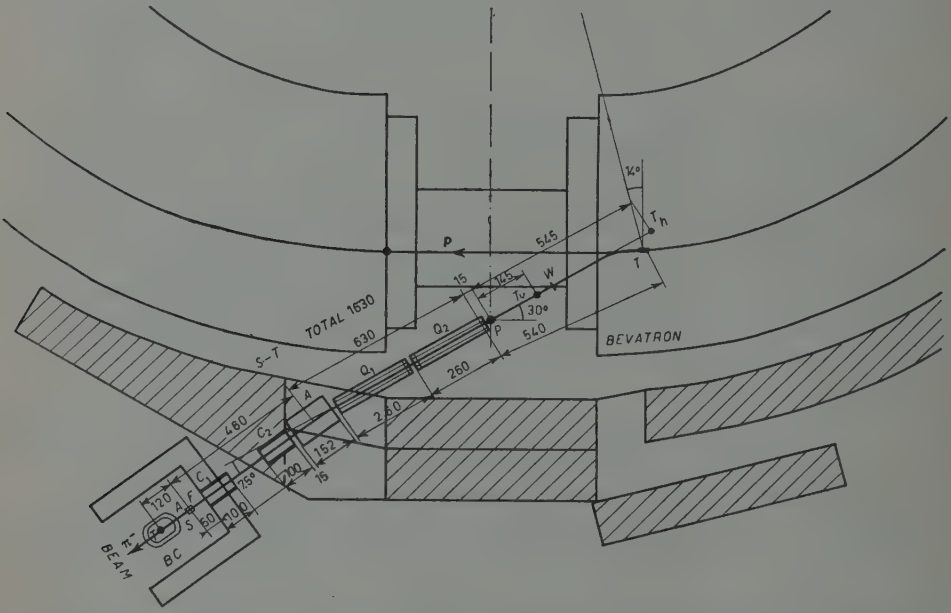

Fig. 1. – Experimental set-up at the Bevatron: numbers are dimensions in cm. *BC* stands for a propane Bubble Chamber, which was exposed before the AF stack.

Measurements in test plates confirmed the focussing effect of the quadrupole lenses and the decreasing intensity with increasing energy. To avoid experi-

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mental complications resulting from edge-effects, a region in the emulsion was chosen where the calculated energy was 5.7 GeV.

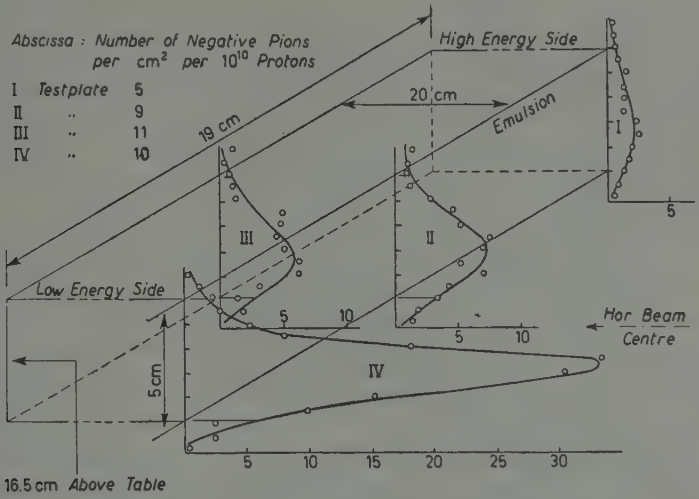

Fig. 2. – Intensity of negative pions versus position in the AF stack as deduced from testplates. The stack was irradiated by collision products of a number of 10^{14} protons hitting the target. The curves I and IV represent the intensity along the corner lines, the curves II and III represent the intensity in the middle of the stack, respectively at the front and at the back. The intensities as deduced from the testplates are in agreement with measurements performed in the AF stack.

The experimental situation is shown in the Figs. 1 and 2.

2. – Anti-hyperon production by pions.

Treating the problem of the production of particles [1] by means of kinematical considerations, the interaction of the pions with nuclei as a complete unit can be ignored, the de Broglie wavelength for a 5,7 GeV pion being $2.1\cdot10^{-14}$ cm. It is found then that by the impact of a 5.7 GeV π^- on a nucleon, antihyperons can be created. This can be seen from Fig. 3 where the thresholds for different reactions are indicated by arrows. The graphs represent the relative probability per energy interval of occurrence of a specified total energy E in the c.m. versus E. The calculations were made assuming the nucleons to be a completely degenerate Fermi gas and accepting a value of the nuclear radius of $1.4\cdot10^{-13}A^{\frac{1}{3}}$ cm.

As can be seen from Fig. 3 antihyperons can be produced in our emulsion. A $\overline{\Lambda}{}^0$ might be produced from proton-nucleon collisions due to the primary Bevatron beam. However, such a $\overline{\Lambda}{}^0$ will probably have decayed before reaching

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a detecting instrument. Having established the possibility of the production of anti-hyperons in emulsion, the detection probabilities in this emulsion have to be considered.

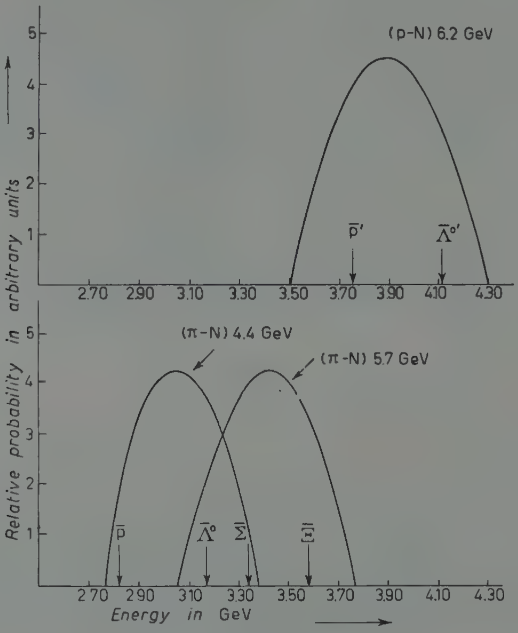

Fig. 3. – Relative probability of occurrence per energy interval of the total energy E in the c.m.s. as a function of that energy for the collision: pion, nucleon (π-N) and proton, nucleon (p-N). The curves are not normalized. The following abbreviations are used:

$\bar{p}$	is the	threshold energy	of	$\pi^- + p \rightarrow n + p + \bar{p}$
$\bar{\Lambda}^0$	»	»	»	$\pi^- + p \rightarrow n + \Lambda^0 + \bar{\Lambda}^0$
$\bar{\Sigma}$	»	»	»	$\pi^- + p \rightarrow n + \Sigma + \bar{\Sigma}$
$\bar{\Xi}$	»	»	»	$\pi^- + p \rightarrow n + \Xi + \bar{\Xi}$
$\bar{p}'$	»	»	»	$p + p \rightarrow p + p + p + \bar{p}$
$\bar{\Lambda}^{0\prime}$	»	»	»	$p + p \rightarrow p + p + p + \Lambda^0 + \bar{\Lambda}^0$

The $\bar{\Sigma}^-$ could be produced at a somewhat lower energy by $\pi^- + p \rightarrow p + \Lambda^0 + \bar{\Sigma}^-$. The introduced notation $\bar{\Sigma}^-$ means the antiparticle of Σ^+.

As little energy is left after the production process, the anti-particle will generally have a velocity corresponding to the velocity of the c.m.s. (0.86 c). This results in a high energy in the laboratory system and therefore the subsequent decay of the anti-hyperon will not easily be detected. The Q-value of the decay is too small to alter the detection probability appreciably. There are however situations where the energy is divided favourably and a charged Σ might stop after a traversal of about 12 cm emulsion. The following decay reactions are of interest:

$$\overline{\Lambda}^0 \rightarrow \overline{p} + \pi^+, \quad \text{(the neutral decay mode not being detectable in emulsion)} \tag{1}$$

$$\overline{\Sigma}^+ \rightarrow \overline{n} + \pi^+, \tag{2}$$

$$\overline{\Sigma}^- \rightarrow \overline{n} + \pi^-, \tag{3a}$$

$$\overline{\Sigma}^- \rightarrow \overline{p} + \pi^0, \tag{3b}$$

$$\overline{\Xi}^+ \rightarrow \overline{\Lambda}^0 + \pi^+ : \tag{4}$$

As in the case with the Σ^0 the $\overline{\Sigma}^0$ will probably not be detected in emulsion.

Reaction (1) seems the most promising one. It might enable the identification of the two charged decay products and give conclusive evidence for the antimatter character of the original particle.

There is little chance to distinguish between reaction (2) and the ordinary hyperon decay,

$$\Sigma^+ \rightarrow n + \pi^+ .$$

In both cases the pion only is observed. The different energies resulting from the expected mass difference of Σ^+ and $\overline{\Sigma}^+$ will probably be masked by the experimental errors as the decay generally takes place in flight.

Analogous arguments hold for reaction (3*a*).

The antiproton of reaction (3*b*) however might be observed by the star produced at annihilation. The event might be reconstructed then by tracing back the antiproton.

Reaction (4) can be identified using observable data only, but the definite proof will be lacking because the antimatter is carried away with the non observable uncharged $\overline{\Lambda}^0$.

At present we have in progress a program for finding stars initiated by antiprotons. Reaction (1) can of course also be detected by tracing back the π^+. So far we have not found any example of the above mentioned reactions. However the detection of a $\overline{\Lambda}^0$ produced in emulsion by a π^- was announced recently [2]. It was decided to start area scanning at the high energy side of the stack at a position where the beam had travelled 15 cm in the emulsion.

3. – Total cross section for high energy pions.

Another interesting quantity that might come out of area scanning is the mean free path for star production of high energy pions in the emulsion.

This quantity has been measured at various energies because it is of interest to know the influence of energy on the mean free path. In principle one has to record the interactions inside a certain region (area scanning of stars) and to know the number of incoming pions in that region. There are however experimental traps.

First of all it is obligatory to find all the relevant data. As the results indicate the human being is not well suited to find all events with area scanning, even if the scanning is done twice. If a correction is applied for stars with only minimum ionizing prongs, which are easily overlooked, one is still faced with discrepancies, as can be seen by comparing the results of area scanning and track scanning (Table I). The still pertaining difference is thought to result from the fact that even stars with black prongs have no equal probability to be found.

The counting of the incoming pions by different observers revealed that this number could be determined accurately.

Another problem one has to solve concerns the decision which tracks are to be ascribed to the pion beam. After preliminary orientation the decision was made to allow a 2° deviation from the central direction.

A different experimental approach which was started as well, is track scanning. This involves the picking up of an incoming pion and following it through the emulsion. This is a more direct and more reliable determination of the interaction mean free path. This method also enables the correction to the area scanning data due to the stars with only minimum ionization prongs. Both methods are to be corrected for the decay of the pion, which results from the travelling time between target and emulsion. This correction amounts to 5%.

With track scanning one has no bias for small stars of minimum ionization tracks. From track scanning data one can therefore infer a correction for the area scanning. Applying all these corrections we obtain the preliminary result collected in Table I.

TABLE I. *Comparison of the results from area scanning (corrected for the loss of stars with only minimum ionization prongs) with track scanning.*

Energy	Area scanning	Track scanning
Mean free path for star production of 5.7 GeV pions in emulsion	(57 ± 6) cm	(41 ± 6) cm

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Errors quoted are from statistics only. In Table II the value of the area scanning is disregarded while the value of track scanning is compared to those of other workers.

TABLE II. – *Mean free path for star production in* G-5 *emulsion by means of negative pions.*

Energy (GeV)	Mean free path (cm)	References
0.75	38.5 ± 2.2	[3]
3.0	35.5 ± 5	[4]
4.2	38.7 ± 3.5	[5]
4.3	36 ± 5	[6]
5.7	41 ± 6	present work

For comparison the value of the geometrical mean free path is 28 cm. One must keep in mind however, that single scattering and charge exchange scattering, are not included in the values given in the above table. We found in 24.3 m of corrected pion track one charge exchange and six single scatters (projected angle $> 1°$). As one sees, within the experimental errors no variation of the mean free path is observed with increasing pion energy up to 5.7 GeV.

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REFERENCES

[1] O. CHAMBERLAIN, W. W. CHUPP, G. GOLDHABER, E. SEGRÉ, C. WIEGAND, E. AMALDI, G. BARONI, C. CASTAGNOLI, C. FRANZINETTI and A. MANFREDINI: *Nuovo Cimento*, **3**, 447 (1956).
[2] D. J. PROWSE and M. BALDO-CEOLIN: *Nuovo Cimento*, **10**, 635 (1958).
[3] M. BLAU and A. R. OLIVER: *Phys. Rev.*, **102**, 489 (1956).
[4] M. SCHEIN, D. M. HASKIN and R. G. GLASSER: *Nuovo Cimento*, **3**, 131 (1956).
[5] J. O. CLARKE and J. V. MAJOR: *Phil. Mag.*, **2**, 37 (1957).
[6] A. MARQUES, N. MARGEM and G. A. B. GARNIER: *Nuovo Cimento*, **5**, 291 (1957).

SUPPLEMENTO AL VOLUME XIV, SERIE X
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N. 2, 1959
4° Trimestre

Origin of Cosmic Radiation.

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1. – Introduction.

The properties of cosmic ray particles in the neighbourhood of the earth have been studied mainly with the help of balloon borne particle detectors. Information on cosmic ray particles in other, more distant parts of the universe comes from astronomical, expecially radio astronomical observations. One measures the electromagnetic radiation emitted by energetic electrons moving in magnetic fields (synchrotron radiation). Nuclei of comparable energy radiate less energy than electronsby a factor $(M/m_e)^4$ and are therefore not observable with these techniques.

The data on cosmic ray particles which have been collected by these methods form the basis of a theory, developed mainly by ŠKLOVSKIJ [1, 2] and GINZBURG [3]. This theory deals with the origin of radio noise as well as with the origin, the acceleration and the propagation of cosmic radiation. It covers, therefore a much wider field than earlier theories on cosmic ray origin. It is not yet complete and important experimental and theoretical questions remain to be discussed. Nevertheless, it has had some remarkable success in correlating experimental data and does not seem to have encountered serious obstacles so far.

In its barest outline the theory may be presented as follows: about once every hundred years a stellar explosion occurs in our galaxy which is so spectacular that during its brief period of highest luminosity the star is designated as a super nova. Less spectacular events, so-called novae explosions, occur at the rate of about a hundred per year, and are, therefore about ten thousand times more frequent.

An explosion leads to the expulsion of a large amount of gas, which subsequently expands into the surrounding space with a velocity of the order of 1000 km/s and breaks up into turbulent elements. The turbulence of the

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(*) At present at the Institute for Theoretical Physics, University of Copenhagen. The Author takes pleasure in expressing his gratitude to professors N. BOHR, A. BOHR and J. K. BØGGILD for their hospitality.

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resulting clouds leads to the build-up of magnetic fields. Ionized particles moving in the clouds can be accelerated in various ways, mainly perhaps in collisions with the resulting magnetic inhomogeneities by a statistical mechanism originally discussed by FERMI [4, 5].

When the gas envelope has expanded sufficiently so that magnetic field strength and density approach that of the surrounding inter-stellar space, the accelerated particles begin to escape into the main galactic volume. Here, their direction is rendered isotropic by further collisions with magnetic inhomogeneities, which now however are too rare to lead to an appreciable further increase in particle energy.

In the interstellar space, the particles suffer energy losses due to various processes. Electrons lose energy through ionization of the interstellar medium, through bremsstrahlung in collisions with atoms, through collisions with thermal photons (inverse Compton effect) and through synchrotron radiation. The nuclear component loses energy through ionization and nuclear interactions. The nuclear collisions lead to meson production and, thereby, to the creation of new high energy electrons and positrons through π-μ-e decay. In the case of complex nuclei, the collisions lead also to spallation reactions and thereby, to a degradation of the charge spectrum and washing out of its structural details as well as to a general shift towards lighter elements.

This picture has been developed quantitatively so that one can attempt an interpretation of:

- — the intensity, frequency spectrum and polarization of the optical and radio emission from the remnants of super nova explosions;
- — the properties of radio emission from other parts of the galaxy and
- — the intensity, energy spectrum and composition of the primary cosmic radiation.

The strength of the theory, as compared to earlier attempts, lies mainly in the fact that it successfully links astrophysical and cosmic ray observations.

In discussing the origin of electromagnetic radiation emitted by super novae and the origin of galactic radio noise, we shall follow GINZBURG and ŠKLOVSKIJ rather closely. Some of their arguments have also been used in the later discussions dealing with the nuclear component of cosmic radiation.

2. – The origin of radio noise.

Identification of radio emission and of strongly polarized light with the radiation emitted by relativistic electrons moving in magnetic fields is based on the following evidence:

a) The angular coordinates of historical super novae outbursts in our galaxy coincide with centers of strong emission of radio noise.

b) One of the strongest emitters, the Crab Nebula (super nova of 1054 A.D.), is at the same time a strong source of radiation in the optical region The light which it emits has quite different properties from the light emitted by hot gases. It is so intense that, if it were due to atomic transitions, it would require an expelled gas cloud weighing more that ten solar masses, which seems unlikely. It exhibits a continuous spectrum with no spectral lines. The dependence on wavelength is more adequately represented by a power law than by Planck's radiation formula. Finally, the light is polarized and, as one increases the resolution and looks at smaller and smaller elements of the turbulent cloud, the degree of polarization increases. It seems, therefore, that polarization is nearly complete, and that failure to observe very strong polarization is due only to the fact that the direction of the polarization vector varies for different parts of the nebula.

On the strength of this evidence, there can hardly be any doubt that the radiation is due to the motion of electrons spiraling in magnetic fields.

Electromagnetic theory permits to calculate the number and energy of electrons giving rise to such radiation.

A simple calculation for the Crab Nebula may be useful because it is the essence of the theory that one can establish, without arbitrariness or serious ambiguities, a relation between the number and the energy spectrum of electrons and the intensity and frequency spectrum of synchroton radiation.

An electron spiraling in a magnetic field emits the greatest power at a frequency,

$$\nu_{\max} = \frac{1}{4\pi}\left(\frac{eH}{mc}\right)\left(\frac{E}{mc^2}\right)^2 = 1.4\cdot 10^6\, H\left(\frac{E}{mc^2}\right)^2 \text{ Hz}\,.$$

Here E is the energy of the electron and H is the magnetic field component perpendicular to its direction of motion.

The energy radiated in various frequency intervals is given by

$$P(\nu, E)\,\mathrm{d}\nu = 16\, eH\left(\frac{e^2}{mc^2}\right) p\left(\frac{\nu}{2\nu_{\max}}\right)\mathrm{d}\nu\,.$$

$p(x)$ is a known function of frequency with a fairly sharp maximum at $x = 0.5$.

The total energy radiated by an electron per unit time is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int P(\nu, E)\,\mathrm{d}\nu = \frac{2c}{3}\left(\frac{e^2}{mc^2}\right)^2 H^2\left(\frac{E}{mc^2}\right)^2 = 0.98\cdot 10^{-3}\, H^2\left(\frac{E}{mc^2}\right)^2 \text{ eV/s}\,.$$

This means that if synchroton radiation losses dominate, it will have radiated half of its energy in a time

$$T_{\frac{1}{2}} = \left[\frac{2}{3c}\left(\frac{eH}{mc}\right)^2\left(\frac{e^2}{mc^2}\right)\left(\frac{E}{mc^2}\right)\right]^{-1} = \frac{16.7}{H^2}\,\frac{mc^2}{E}\ \text{years}\,.$$

We now apply these formulae to the Crab Nebula. The Crab Nebula is at a distance $R = 4200$ light years from the solar system. The energy received on the earth has been determined accurately at $\lambda = 4300$ Å and is $F_\lambda = 2\cdot 10^{-23}$ erg/cm² s (Hz). If one then assumes that all electrons have just the correct energy to give their maximum radiation at $\lambda = 4300$ Å, their number must be

$$N_e = \frac{4\pi R^2 F_\lambda}{P(\nu_{\max}, E)} = \frac{2\cdot 10^{43}}{H}\ \text{electrons}\,,$$

with an energy

$$E_e = mc^2\left[\frac{4\pi c}{\lambda}\left(\frac{mc}{eH}\right)\right]^{\frac{1}{2}} = \frac{10^{10}}{\sqrt{H}}\ \text{eV}\,.$$

Assume that the energy radiated by the Nebula was not smaller in the past than it is today; then if we choose $H > 10^{-4}$ G, electrons which began radiating 900 years ago, when the Crab Nebula was born, have already lost at least half of their energy. The total energy given to these relativistic electrons since the super nova explosion is then of the order of

$$W_e = W_{e\,\min} = N_e E_e \frac{900}{T_{\frac{1}{2}}} = 3.5\cdot 10^{47}\ \text{erg}\,,$$

independent of the field strength H.

If, on the other hand, we assume $H < 10^{-4}$ G, then the electrons accelerated in the earliest stages of the super nova are still radiating and the energy of the relativistic electrons necessary to produce the observed light emission will be larger, namely

$$W_e = N_e E_e = W_{e\,\min}\left(\frac{10^{-4}}{H}\right)^{\frac{3}{2}}.$$

Thus, a resonably conservative estimate of the number and energy of relativistic electrons and of the field H of the Crab Nebula is

$$N_e = 2\cdot 10^{47}\ \text{electrons}\,,$$

$$E_e = 10^{12}\ \text{eV}.$$

$$W_e = 3\cdot 10^{47}\ \text{erg}\,,$$

$$H_e = 10^{-4}\ \text{G}\,.$$

A similar calculation in the radio band shows that there radiation must be due to about $N = 10^{50}$ electrons of energy $E \approx 5 \cdot 10^8$ eV or a total energy of $W \approx 10^{47}$ erg. For the radio source in Cassiopaeia A (super nova of 369 A.D.) which is the strongest in our galaxy, one obtains an energy estimate which is many times larger than for the Crab Nebula. $N_e = 10^{52}$ electrons above $5 \cdot 10^8$ eV and $W_{e\,\min} \approx 10^{49}$ erg. Such calculations can be refined somewhat, but the results do not differ appreciably.

One can also calculate the radiation spectrum which would be produced by electrons with a given energy distribution, f.i.; the power law

$$N_e(E)\,\mathrm{d}E = \frac{K_e\,\mathrm{d}E}{E^\gamma}\,.$$

It is easily shown that this leads to a radiation spectrum of the form

$$I_\nu\,\mathrm{d}\nu = \frac{K_\nu}{\nu^{(\gamma-1)/2}}\,\mathrm{d}\nu\,.$$

In the case of the Crab Nebula, the radiation spectrum depends on the frequency as $\nu^{-0.3}$ and requires, therefore, an electron spectrum proportional to $E^{-1.6}$. This is flatter than the particle spectrum of cosmic radiation observed on the earth. In Cassiopaeia A, $I_\nu \sim \nu^{-0.8}$, and the electron spectrum is of the form $E^{-2.6}$ which closely resembles the energy spectrum of the nuclear component observed at the top of the atmosphere.

Apart from the strong point sources of radio noise associated with expanding envelopes of super nova explosions, there is a general background of radio noise in our galaxy which outside of the galactic plane depends only weakly on coordinates and is produced in an approximately spherical system of radius $\sim 50\,000$ light years surrounding the center of the galaxy. At a wave length of a few meters, therefore, the size and shape of our galaxy is very different from the flat disc-shaped spiral observed in visible light. A similar situation exists in $M31$, a spiral galaxy resembling our own and also in other galaxies of the same type. From these observations one may conclude that relativistic electrons as well as gas clouds and magnetic fields exist in a spherical volume surrounding the galactic disc.

A cloud moving in the known gravitational field produced by the galactic mass which is mainly concentrated in the galactic disc cannot reach such a great distance from the galactic plane unless its velocity u is of the order of or larger than 100 km/s. The pressure exerted by the gases must be counteracted by the weight of the overlaying gas layer. From this one can evaluate roughly the gas density ϱ. The clouds will also give rise to magnetic fields which contribute to the pressure to be counteracted by gravitation. The energy stored in the field, $H^2/8\pi$, is probably in equilibrium with the kinetic energy of the

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clouds, $(1/2)\,\varrho u^2$, but in any case is not likely to exceed it. From such considerations one can estimate the gas density and magnetic field at various distances from the galactic center. For the halo (*), surrounding the galactic disc ŠKLOVSKIJ [2] assumes:

At a radius $r < 30\,000$ ly	$\varrho = 10^{-2}$ atoms/cm^3	$H = 6\cdot10^{-6}$ G
At a radius $r > 30\,000$ ly	$\varrho = 3\cdot10^{-3}$ atoms/cm^3	$H = 3\cdot10^{-6}$ G
In the galactic disc itself one usually assumes	$\varrho = 1$ atoms/cm^3	$H = 6\cdot10^{-6}$ G

Having estimated the strength of the magnetic field, one can now analyse, the radio emission from the halo and finds that the radiation is presumably due to electrons with an energy spectrum of the form

$$N_e(E)\,\mathrm{d}E = \frac{K_e}{E^{-2,6}}\,\mathrm{d}E\,,$$

that the density of electrons is fairly uniform throughout most of this sphere, and that the space density of those electrons whose energy exceeds 0.5 or 1.0 GeV is of the order of 10^{-12} particles/cm^3.

The theory now asserts that the super nova and possibly also nova explosions in the galactic disc supply the energy which is emitted in the form of radio waves by the galactic halo. One argument is based on the fact that the energy injected by super novae is of the same order of magnitude as the output of radio noise. Here one assumes that the energy given to electrons in the average super-nova envelope is at least of the same order as the minimum estimate of $W_e \approx 3\cdot10^{47}$ erg obtained from the Crab Nebula (the energy of the source in Cassiopea A is 100 times greater). One estimates the frequency, ν_{SN}, of super-nova outbursts in our galaxy as one in every hundred years. (This takes into account that a large fraction of the outbursts are hidden from our view by dense dust clouds and could not, therefore, have been detected before the advent of radio astronomy). The particle energy released into the galactic halo must, therefore, be of the order of $W_e\nu_{SN} = 10^{38}$ erg/s. On the other hand the total radio emission of the galactic halo has been estimated within a factor two to be $F = 2.5\cdot10^{38}$ erg/s. Thus it seems that super nova explosions can provide the necessary energy.

ŠKLOVSKIJ presents another interesting argument in favor of the connection between stellar explosions and galactic radio noise (quoted by GINZBURG [6])

(*) The physical conditions which exist inside the galactic halo are not yet known with certainty. (See *Proc. 3rd Symposium on Cosmical Gas Dynamics*, *Rev. Mod. Phys.* **30**, 925, 1958).

The clouds of Magellan, which are small galaxies adjacent to our own, are quite weak emitters of radio noise. Stellar explosions of the Nova type are also much less frequent there than in our galaxy. (No comparison can be made between the rates of super novae outbursts because they are too rare.) This observation gives some support to a theory which links those two phenomena. It also implies that the electrons which radiate within the halo of our galaxy cannot easily escape through its boundary and are, therefore, unable to fill the volume occupied by the clouds of Magellan.

3. – The nuclear component of cosmic radiation.

So far the theory has no connection with the primary cosmic rays observed in the neighbourhood of the earth, except in so far as it suggests more refined experiments: it states that in addition to the primary nuclei which have been observed, there should be primary electrons with energies of about 1 GeV or larger and a density which in our neighbourhood should be about the same as in the rest of the galaxy, namely $\sim 10^{-12}$ particles/cm^3. Cosmic ray measurements up till now can only confirm that the density of high energy electrons is not higher than this value. Primary electrons have not yet been observed; the flux predicted by the theory lies close to the limit of sensitivity of the experiments which have been carried out so far.

Secondly, the theory asserts that the electrons should have a differential energy spectrum proportional to $E^{-2.6}$, very similar to the energy spectrum of primary nuclei, which behaves like $E^{-2.5}$.

But a real connection between the theory of the origin of radio noise and the origin of primary cosmic ray nuclei is established with the help of the following argument: one knows of only two ways in which high energy electrons can come into being, (*a*) they may be the decay products of unstable particles produced in high energy nuclear interactions, or (*b*) they may have been accelerated by some process involving electro-magnetic interactions.

Whether one accepts (*a*) or (*b*) one is led to the conclusion that the creation of high energy electrons implies the existence of high energy nuclei. In case (*a*) this is obvious. In case (*b*) it follows from the fact that wherever there are free electrons there must be ions and it seems difficult to think of mechanisms which will successfully accelerate electrons without accelerating ions.

According to this theory therefore, emission of radio noise implies the presence not only of high energy electrons but also of nuclei with probably still higher energies. These particles are accelerated in super nova- (and perhaps nova-) explosions and are released into the interstellar space when the magnetic field density of the expanding gas envelope has reached the value which it has in the surrounding medium. This may take from 1 000 to 3 000 years. Then the envelope ceases to be a strong radio emitter. The life time of super novae envelopes can be estimated from the number of radio sources which are active in our galaxy to-day and from the rate at which explosions occur in our and similar galaxies.

4. – The primary cosmic radiation observed near the Earth.

Before we go further into the relation between cosmic ray nuclei, super novae and galactic radio emission, it may be useful to summarize very briefly the existing information on primary cosmic radiation in the neighbourhood of the earth. The radiation consists of the nuclei of various elements ranging in atomic weight from hydrogen to iron or nickel (Peters [7]). The energy spectrum follows a power law of the form

$$N_A(E)\,\mathrm{d}E = \frac{K_A}{E^{\gamma}}\,\mathrm{d}E\,,$$

where K_A is a constant for a given element of atomic weight A and E is the total energy of the primary nucleus (including rest mass) (Kaplon *et al.* [8], McDonald [9]). From a kinetic energy of $\frac{1}{2}$ GeV per nucleon to a few thousand GeV per nucleon the exponent γ has the same value namely $\gamma \approx 2.5$ for all components. At higher energies one does not yet know the relative strength of individual primary components, but only the total number of particles as a function of energy. The spectrum continues to be of the form $E^{-\gamma}$ (Lal [10]). Beyond $E \approx 10^4$ GeV the exponent begins to increase slightly and reaches $2.75 \div 3.0$ at $E \approx 10^8$ GeV. The largest air showers observed until now correspond to a primary particle with an energy of at least $2 \cdot 10^9$ GeV (Clark *et al.* [11]). If it was a heavy nucleus its energy per nucleon was probably of the order of a few times 10^7 GeV.

No electrons and no γ-rays have as yet been observed in the primary radiation. At a particular energy, electrons, if present, are known to comprise less than one per cent of the primary flux. (Critchfield *et al.* [12]).

Various processes can lead to changes in the number of cosmic ray particles incident on the earth (Dorman [13]). Disturbances on the sun and the eleven year sun spot cycle which affect the intensity of non-relativistic cosmic ray particles, can occasionally make themselves felt up to energies of about 10 GeV. When the solar surface is quiet, cosmic ray intensity is at its maximum and, therefore, probably approaches the true value which it has in outer space.

The sun itself can occasionally act as an emitter of energetic nuclei. Such outbursts which are connected with large solar flares are however extremely rare.

Primary cosmic rays whose energy exceeds ~ 20 GeV are not affected by solar activity, they show no time variation and are, therefore, completely isotropically distributed in the surrounding space.

Let us now briefly describe the chemical composition of the primary radiation (see PETERS [7], APPA RAO *et al.* [14]). When the comparison is made between particles of equal energy per nucleon the hydrogen-helium ratio of 15 is the same as the corresponding abundance ratio estimated for the universe as a whole. The elements C, N, O and the heavier elements up to iron seem to be five to ten times more abundant with respect to hydrogen in cosmic radiation than they are in the universe. The group of elements Li, Be and B whose universal abundance with respect to hydrogen is low $(3\cdot10^{-9})$ are $\sim 100\,000$ times more abundant in the cosmic radiation. The large discrepancy between the abundance of the last mentioned group of elements in the cosmic radiation and its abundance in the universe as a whole, is connected with collisions of complex primary cosmic ray nuclei in interstellar space. We shall return to this point later.

When the comparison is made between particles of equal kinetic energy (not energy per nucleon), then hydrogen accounts for only half of the intensity, helium for one quarter, the group of light nuclei C, N, O and all elements heavier than oxygen account each for one eighth of the total intensity.

If we confine ourselves to particles whose kinetic energy exceeds 0.5 GeV/nucleon, the particle density of the nucleon component in the neighbourhood of the earth amounts to $1.7\cdot10^{-10}$ nucleons/cm^3 and the energy density to 0.56 eV/cm^3. This energy density is somewhart larger but of the same order of magnitude as that of starlight in our parts of the galaxy.

Information on cosmic radiation in the neighbourhood of the earth is based on nuclear emulsion- and on counter- and ionization chamber measurements carried out near the top of the atmosphere; in the case of the very largest energies, it is based on measurements of large particle showers observed on the ground.

5. – The chemical composition of nuclear primary radiation and its history of motion.

A number of conclusions can be drawn from the chemical composition of the nuclear component which are relevant to the origin of cosmic radiation. The mere fact that the primary radiation contains not only nucleons but also complex nuclei shows, that the particles cannot have traversed a large amount of matter after acceleration, not more, in any case, than corresponds to one or two mean free paths for collision of iron nuclei in interstellar hydro-

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gen. This amounts to perhaps 5 g/cm². A more precise value is obtained from the relative abundance of Li, Be, and B, those light nuclei which are extremely rare in the universal abundance of elements, but which appear in measurable amounts in the primary cosmic radiation. In the upper atmosphere one observes that the ratio of (Li+Be+B) to (C+N+O) is much larger among primary nuclei incident under large zenith angles than it is among those coming from the zenith direction. The ratio increases rapidly with the thickness of the air layer which the particles have traversed before reaching the detector. This gives a clue as to why these light nuclei are present in the primary radiation with a relative abundance which is 100000 times higher than in the universe as a whole. They arise in nuclear collisions by partial break up of the heavier primary component, in particular in collisions of carbon with hydrogen. C-p collisions lead in more than 70 % of all cases to a residue of atomic number $Z = 3,4$ or 5. (The remaining cases are those where the carbon nucleus is completely disintegrated into nucleons and α-particles.)

The ratio (Li+Be+B) to (C+N+O) $\approx 8\,\%$ which is observed at the top of the atmosphere (Appa Rao *et al.* [14]) implies that the nuclei which reach the solar system have traversed only about one half g/cm² of hydrogen (but in any case not more than one g/cm²) since the early stages of acceleration when their energy had reached the few MeV/nucleon necessary to overcome Coulomb repulsion and produce nuclear reactions. The condition that the primary particles have traversed only a very small amount of interstellar matter when they reach the solar system is not easy to satisfy and places strong restrictions on acceptable theories of cosmic ray origin.

Because the amount of matter which has been traversed is small, one can conclude that the relative abundances of the main nuclear components have not changed much during transit through space. The chemical composition of the primary radiation reflects the composition of the source region; the comparatively high percentage of nuclei in the range of atomic numbers between carbon and iron must be a property of this source. The fact that the flux of helium nuclei in the cosmic radiation is of a magnitude to be expected from its universal abundance suggests that the inital acceleration takes place in a region which is sufficiently hot and rarified to insure that helium atoms are completely ionized in spite of their high ionization potential.

One of the most remarkable features of the primary radiation is the fact that all components have the same energy spectrum over a very large range of energies. This is not only true for the heavy particles which, once they have been stripped of electrons, have identical charge to mass ratios and therefore, follow identical trajectories in an electromagnetic field; also the proton component, for which the charge to mass ratio is always higher than for complex nuclei, has the same energy spectrum. The experimental fact, that the energy distribution does not depend on the charge of the nuclei is, of course, a strong restriction on acceptable explanations of the mechanisms for particle acceleration.

A successful theory will have to explain further why, at least in our neighbourhood, electrons are much rarer than nuclei of comparable energy. It must account for the total cosmic ray intensity and suggest an adequate source to supply the necessary energy. Furthermore, it must provide an explanation for the high degree of spatial isotropy in the cosmic ray flux near the earth. Let us see how the super nova theory deals with these problems.

6. – Cosmic ray nuclei and electrons in the galactic halo.

We have already mentioned that the magnetic fields which seem to exist in the galactic halo and in the galactic disc are of the same order of magnitude and that the radio noise emitted from the halo region does not depend strongly on coordinates. Therefore, the density and energy distribution of electrons also should be reasonably constant throughout the main body of the galaxy. This may be assumed to hold also for the nuclear component and one may therefore take for distant regions of the galaxy the same nucleon energy density ($W_N = 0.56\ \text{eV/cm}^3$) which is found near the earth.

An interesting question is whether the electrons, which are found at great distances from the galactic plane, are those which were originally dispersed from the gaseous envelope of super nova explosions and have diffused into the halo, or whether they are locally produced as decay products of mesons resulting from nuclear interactions of primary nuclei in interstellar space.

There are fairly strong arguments in favour of the latter hypothesis the most important being that, as shown below, the energy given to electrons produced as a result of nuclear interactions in the interstellar gas appears to be of the same order of magnitude as the energy radiated by the electrons which are reponsible for the emission of radio noise from the galactic halo. The average time T_p between nuclear interactions of primary protons is

$$T_p = \frac{1}{n\sigma c},$$

where one may take for the gas density $n = 10^{-2}$ hydrogen atoms/cm³ and for the p-p cross section $\sigma = 30\ \text{mb}$. One then finds $T = 10^{17}\ \text{s} = 3\cdot 10^9$ yrs. This is comparable though less than the age of our galaxy estimated as $6\cdot 10^9$ yrs. In a nuclear encounter a high energy proton loses an estimated 30% of its energy to meson production. One third of this transforms into neutral mesons and escapes from the galaxy in the form of γ-rays. Of the remaining two thirds, 26% are transmitted to electrons via π-μ-e decay.

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Heavy nuclei also contribute to this electron component. Their life span between collisions is shorther than that of protons, but the result of first collisions is mainly the breakup of the complex nuclear structures into smaller fragments; the conversion of their energy into mesons does not proceed much faster than in the case of protons.

One may, therefore, estimate the energy input into the electron component to be $P_e = \frac{2}{3} \cdot 0{,}30 \cdot 0.26 \cdot (W_N/T_p)V = 5 \cdot 10^{37}$ erg/s, where $V = 10^{68}$ cm^3 is the estimated volume of the galactic halo.

With the assumptions made for the gas- and magnetic field density in the halo the main energy loss of electrons with more than 0.5 GeV is due to synchrotron radiation. The power input of $5 \cdot 10^{37}$ erg/s can, therefore, be compared directely with the measured power radiated by the galaxy in the form of radio noise. Within a factor two this has a value of $2.5 \cdot 10^{38}$ erg/s. In view of the roughness of the estimates involved, it is striking that the two numbers come out to be of the same order of magnitude.

In addition to calculating the total energy which goes into the electron component, one can also estimate the electron energy spectrum if one is willing to make certain simplifying assumptions about the meson production process.

We shall assume that in the energy interval of interest, the degree of inelasticity, that is the fraction of the available energy which goes into meson production, remains constant. We further assume that in this interval the multiplicity of meson production varies with primary energy as $E^{\frac{1}{4}}$. The average energy of the mesons and their decay products varies then as $E^{\frac{3}{4}}$ and a differential nucleon spectrum proportional to $E_N^{-\gamma}$ gives rise to an electron spectrum proportional to $E_e^{-\alpha}$, where $\alpha = \frac{4}{3}\gamma - \frac{2}{3}$. Using the experimental exponent $\gamma = 2.5$ for the nuclei, one obtains $\alpha = 2.67$ in close agreement with the value 2.64 derived from the wavelength dependence of galactic radio emission. Thus, it seems that an electron density, which is one hundred times smaller than the nucleon density and has nearly the same energy dependence, *could* be produced by collisions of cosmic ray nuclei with interstellar hydrogen and *could* give rise to a galactic radio emission with intensity and wavelength dependence in agreement with what is being observed.

There is, however, an unresolved difficulty with this picture. If one continually injects electrons with an energy spectrum proportional to $E_e^{-\alpha}$ and if these electrons are subject to a rate of energy loss proportional to E_e^2 (as they are when synchrotron radiation dominates other losses), then a steady state spectrum is established with an exponent $\alpha + 1$. The time necessary for reaching this steady state is of order $T_{\frac{1}{2}}$ which under the conditions prevailing in the galactic halo lies between 10^8 and 10^9 years. One should, therefore, expect an electron spectrum in the halo which is steeper than the spectrum of cosmic ray nuclei assumed to be responsible for their production. This difficulty has not been resolved. The question as to whether the cosmic ray

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electrons in the halo are accelerated together with the nuclear component and get there by diffusion, or whether they are secondary to the nuclear component cannot yet, therefore, be answered with confidence.

7. – The energy requirement.

The total energy to be supplied by the super nova and nova mechanism should be compared with the rate P_N at which cosmic ray nuclei lose energy to the charged and neutral meson component. Using the same constants as before one finds $P_N = 2.6 \cdot 10^{38}$ erg/s. Assuming a frequency of super novae outbursts, $\nu_{SN} = 10^{-2}$/year $= 3 \cdot 10^{-10}\ s^{-1}$, the average super nova should supply $P_N/\nu_{SN} = 10^{48}$ erg. This is approximately equal to the energy estimate which we have made for the high energy electrons in the Crab Nebula.

Of course, one has no right to assume that super novae outbursts in our galaxy have occured at a constant rate or that the size of the galaxy and its field and matter density have remained unchanged for a period of length T'_p. The estimates are made here only to show that, so far, no violent discrepancies in orders of magnitude arise from the postulates of the theory.

8. – The diffusion of cosmic ray nuclei in the galaxy.

Next, one may consider how this picture of cosmic ray nuclei continuously colliding with protons of the interstellar gas can be reconciled with the fact that the nuclei observed in our neighbourhood have not suffered many collisions and have not traversed more than one g/cm² of hydrogen before arriving.

GINZBURG [3] has discussed this problem. He has used a model in which the turbulent magnetic clouds are treated as scattering centers, and the motion of cosmic ray particles is described as a spherically symmetric radial diffusion of particles outwards from the galactic center where most of the super novae explosions are assumed to occur. In such a model the chemical composition of primaries near the galactic center will be that of the source, and the larger the distance from the center, the older the particles, which means that they have on the average traversed a greater amount of gas and suffered a greater number of spallation producing collisions. GINZBURG has tried to show that it is just possible, by choosing a rather large distance $L\,(3 \cdot 10^{20}$ cm) between scattering centers, to keep the ratio (Li+Be+B) to (C+N+O) below 10% at a distance from the galactic center corresponding to that of the solar system. Within errors, this would still be consistent with the composition observed for the particles incident on the atmosphere.

One can, however, raise objections to this calculation. Firstly, the yield of Li, Be, B fragments in collisions of heavier primaries with protons is likely to be considerably larger than assumed by GINZBURG; the values used by him were derived from experiments involving the fragmentation of nuclei in collisions with complex target nuclei rather than with hydrogen. The second serious objection is that a model of diffusion which demands irregular and randomly oriented magnetic fields is not applicable to the motion of particles in the galactic disc, where apparently the fields are not chaotic but are oriented along the spiral arms. Since the solar system is located inside a spiral arm, the particles which reach it from the direction of the galactic center must spiral along those lines of force. The resultant anisotropy in the motion of particles in the galactic disc is reflected in an anisotropy of radio noise, as was recently pointed out by TUNMER [15]; it gives rise to a bright belt of radio emission which is perpendicular to the galactic plane and to the direction of our spiral arms and seemingly passes through the sun.

If one assumes that cosmic ray particles reach the solar system by traveling along our spiral arm, there is no longer any difficulty with keeping the amount of matter traversed sufficiently small to account for the scarcity of fragmentation products among the complex primary nuclei.

Assume an isotropic emission of particles from the super nova envelope. Then the average amount of matter traversed by a particle traveling a distance L along the spiral arm is given by

$$\langle x \rangle = \varrho L \int_{\cos\theta=\cos\theta_{\max}}^{1} \frac{\mathrm{d}\cos\theta}{\cos\theta},$$

where θ is the pitch angle of the spiral motion. It is reasonable to assume that the drift velocity of the particles, $c\cos\theta$, can, except for brief periods, not be smaller than the velocity u with which clouds of gas and resultant distortions of magnetic fields travel in the galactic plane. One can then write $\cos\theta_{\max} = u/c$ and

$$\langle x \rangle = \varrho L \log\frac{c}{u}.$$

Our distance from the galactic center $L = 25\,000$ light yrs. The matter density in the disc is usually taken as $\varrho = 10^{-24}$ g/cm³, and the velocity of clouds as $u = 5$ km/s. This leads to $x = 0{,}25$ g/cm², which is less than the maximum permissible value of 0.5 to 1.0 g/cm².

There is, therefore, no inconsistency in assuming that the particles which one observes entering the earth's atmosphere originate near the center of the galactic plane and travel along the spiral arms where the magnetic field is known to be fairly homogeneous.

The spiral motion automatically insures an isotropic distribution of particles in the plane perpendicular to the spiral arm. It does not, however, provide symmetry in the forward and backward directions along the arms. The symmetry which one observes in the primary radiation which enters the atmosphere must be the effect of a reflection of particles by magnetized gas clouds presumably in the planetary system. There is evidence that such reflections of particles in the neighbourhood of the sun do, in fact, take place. On February 23, 1956 a very large flare occurred on the sun and a large burst of nuclei, accelerated at the sun, struck the earth. Most of the particles were of low energy, but there were some with an energy as high as 50 GeV. The particles continued to arrive for many hours after the flare had subsided. While the flux initially was highly anisotropic, a tendency for reaching isotropy was observable, as soon as the visible flare had disappeared and complete isotropy was established soon thereafter (Dorman, [13] Lüst and Simpson, [16]). This must have been due to reflections from magnetic inhomogeneities inside the planetary system located within a few, light minutes from the sun. If a similar configuration of magnetic fields extends over a much larger distance, particles with correspondingly higher energy will be rendered isotropic by the same mechanism. If the magnetically disturbed region surrounding the sun extends to a distance of the order of a light year, the observed isotropy could be locally produced even for particles whose energy is great enough to initiate very large air showers.

In the picture outlined here the isotropy which one observes for the cosmic ray particles entering the earth's atmosphere is essentially a local phenomenon and is due to the fact that we are located in the neighbourhood of a star which ejects ionized gas clouds and therefore surrounds itself with turbulent magnetic fields. In the space between stars the radiation in the spiral arm may not be isotropic and the distribution of radio noise is an indication of that. Of course particles which reach the solar system may already have suffered some reversals of direction in the spiral arm before arrival. There could not however have been more than two or three such reversals, otherwise, the amount of gas traversed by particles before arrival would have to exceed the permissible limit derived from the charge composition of primary nuclei.

Once the particles leave the spiral arm and enter the galactic halo the fields are probably more randomly oriented and in that region the motion of particles can perhaps be adequately described as diffusion. We have seen that it will take a time of order T_p, which is of the order of the age of the galaxy, before this halo is filled to its saturation value with cosmic ray nuclei. A large fraction of the older particles will have already suffered collisions with a resultant change in atomic weight and atomic number. Presumably, therefore, the galactic halo contains fewer complex nuclei than are to be found in the neighbourhood of the earth.

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9. – Conditions at the galactic boundary.

It is interesting to consider the question whether particles which approach the boundary of the galactic halo, are efficiently reflected, or whether they can escape into extragalactic space. If the magnetic fields arise entirely from a system of currents within our own galaxy, the magnetic lines of force are closed and reflection should be very efficient. If, on the other hand, there are large extragalactic currents so that our galaxy is magnetically linked with other systems, particles will be able to escape and travel beyond the galactic boundary. We have already mentioned that the low level of radio noise in the clouds of Magellan speaks in favor of the retention of particles within the boundaries of the galactic halo. Apart from that, there is no strong argument at present which has a bearing on this question. It is true that if the loss of particles through the boundary were appreciable, the power output of the sources of cosmic radiation would have to be correspondingly higher. It must, however, be remembered that the loss of particles through the boundary does not only depend on the number of magnetic lines of force leaving our galaxy. It also depends on the diffusion length in the halo which means effectively on the mean distance between magnetic inhomogeneities which act as scattering centers. If the diffusion length is very small, particles will in general lose their energy by nuclear collisions before reaching the boundary. Existing estimates, though not very reliable, indicate that the diffusion length is probably sufficiently small compared to the radius of the galactic halo so that diffusion length rather than the reflectivity at the boundary determines the space density of cosmic radiation in the galactic halo.

10. – Cosmic ray nuclei in super nova envelopes.

We shall now briefly discuss conditions for particle acceleration in a super nova. One of the first questions one can ask is. Does the acceleration occur during the explosion itself or at some later stage by a mechanism which operates in the expanding gas envelope?

A simple calculation shows that the first alternative would lead to contradiction with experiment: if the acceleration had taken place during the explosion or during the earliest phase of the gas expansion, the amount of matter traversed by the accelerated particles would have to exceed considerably the upper limit of 1 g/cm² set by the relative abundance of Li, Be and B in the primary cosmic radiation. Suppose one considers the Crab nebula as typical and follows the astrophysical estimate that the gas which was expelled by its central star has a mass of approximately 0.1 solar masses.

Then the gas density in the envelope is

$$\varrho(t) = \frac{0.1\,M_0}{\frac{4}{3}\pi(R_0 + vt)^3},$$

where v is the velocity of expansion and R_0 the radius which the envelope had when acceleration began. The composition of the nuclear component requires that x (the amount of matter traversed by the particles during the expansion of the envelope) is less than 1 gm/cm². Therefore

$$x = \int_0^T \varrho c\,\mathrm{d}t \leqslant 1\ \mathrm{g/cm^2},$$

where T is the time at which particles escape and the envelope dissolves into the outer space. One finds that only small values of t can contribute appreciably to the integral and that therefore the result does not critically depend on the duration T of the expansion process. One can write, therefore

$$R_0 \geqslant \left[\frac{0.1\,M_0}{\frac{8}{3}\pi(v/c)x}\right]^{\frac{1}{2}} \approx 6\cdot 10^{16}\ \mathrm{cm}\,.$$

This is 4000 times larger than the distance from the sun to the earth and implies that the cosmic ray nuclei have been accelerated several decades after the original explosion.

Similarly, one can show that in the earliest stages of super novae activity the energy loss of electrons due to the inverse Compton effect will be so large as to effectively prevent their acceleration.

The problem is, therefore, to find an accelerating mechanism which will work long after the expulsion of the gas. A very attractive hypothesis originally proposed by FERMI [4], is that particles are accelerated by a statical mechanism. The nuclei collide with magnetic clouds which move with the velocity u. The particles are reflected provided the clouds are larger than the particle's radius of curvature. There is a tendency to establish a Boltzmann distribution and to reach a state of equipartition of energy between the particles and the enormously more massive clouds. This tends to increase the particle energy at the expense of the clouds. The process is truly statical in the sense that it does not depend on the details of the interaction, f.i. it does not depend on the strength of the field H, nor on the electric charge Ze carried by the particles. On the average the increase in energy per collision amounts to

$$\Delta E = \frac{u^2}{c^2}\,E,$$

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where E is the particle energy including rest mass. After n collisions the energy becomes:

$$E = Mc^2 \exp\left[\frac{u^2}{c^2}\right] n \, .$$

If the entire process lasts for a time T, the probability that a particle will have undergone between n and $n+\mathrm{d}n$ collisions is given by

$$W(n)\,\mathrm{d}n = \frac{\tau}{T} \exp\left[\frac{-n\tau}{T}\right] \mathrm{d}n \, ,$$

where $\tau = L/v$ is the mean time interval between such collisions. v is the particle velocity and L is the average distance between the clouds. This leads to an energy spectrum

$$W(E)\,\mathrm{d}(E) = \frac{K\,\mathrm{d}E}{E^\gamma}; \qquad \text{where} \qquad \gamma = 1 + \frac{c^2 L}{u^2 T v} \approx 1 + \frac{cL}{u^2 T} \, .$$

The mechanism was originally supposed to be operative in the interstellar space. But there the cloud velocities u seem to be too small and the distance between clouds L too large to provide a sufficient rate of acceleration. It may, however, be operative in the envelope of super novae. It would satisfy important requirements imposed by observation:

a) It leads to a power spectrum whose exponent is independent of the nature of the particle, in particular of its electric charge and of the charge to mass ratio.

b) It is effective in any turbulant gas cloud and does not require critically adjusted configurations of fields.

c) The physical constants appropriate to an expanding super nova envelope are such that they *could* lead to the observed exponent of the power spectrum.

d) The constants are such as to make possible the acceleration of nuclei at a rate which is larger than their rate of energy loss even if a particle carries a high nuclear charge and starts out at a fairly low energy.

e) The acceleration is likely the last long enough to permit some nuclei to reach the highest energies observed in cosmic radiation; the dimensions and magnetic fields in the envelope could be adequate to retain such particles provided they are heavy nuclei so that their momentum per nucleon does not have to exceed a few times 10^{16} eV/c.

There are, however, some difficulties with this picture. One is, that the exponent of the power law is very sensitive to the value of not very simply related quantities. The life span of the super nova envelope, T, the velocity, u, and the size L of the turbulent elements. While it is possible that in a particular envelope these constants combine in such a way as to give a value for γ which is close to the experimentally determined exponent, there seems to be no obvious reason why different super novae should not have values of γ, and, therefore, radio frequency spectra, which differ very considerably from each other. Not many centers of radio emission have been observed so far, but those which have been investigated, suggest a rather narrow range of values, γ, lying between 1.5 and 3.

More serious perhaps is the problem of the ratio between energy stored in the electron component and energy stored in the nuclear component. When operating for a fixed period the Fermi mechanism accelerates particles to an energy proportional to their rest mass. The energy in the nuclear component should, therefore, be circa 2 000 times larger than that in the electron component. If one takes into account that electrons may be handicapped because of the many mechanisms by which they can lose energy in the interval between accelerating collisions, the ratio becomes still larger. An energy which is 2 000 times higher than that present in relativistic electrons would exceed the entire kinetic energy of the super nova envelope and could, therefore, not possibly be transmitted to the nuclear component by a Fermi mechanism.

One might think that the ratio of nuclei to electrons could be reduced by assuming that while nuclei and also some electrons are accelerated by the Fermi mechanism, the majority of the electrons in the super nova envelope have not been accelerated in this manner but are the result of nuclear interactions. The ratio between high energy electrons and high energy nuclei could then be determined in a way analogous to that which was discussed in connection with the electrons in the galactic halo. This, however, leads to another difficulty. The chemical composition of the nuclear component would be greatly altered if one permitted more than 2% of the protons and a correspondingly larger fraction of complex nuclei to collide while moving inside the super nova envelope. It is, therefore, not possible to obtain in this manner a lower ratio of nuclei to electrons without at the same time altering the chemical composition of the nuclear component in a way which is not consistent with observation.

One is apparently forced, either to abandon the Fermi mechanism and assume that most of the energy that appears in relativistic particles comes from the star itself rather than from the kinetic energy of the gases in the envelope, or else to assume that one has overlooked an important feature in the accelerating process which favors electrons over protons. Such a feature may be the fact that β-decay of radioactive nuclei provides a strong source of electrons which are already relativistic and will therefore, gain energy more rapidly than they would if they had to start at low energy.

It would be most important to discover the electron component in the primary radiation and to measure the sign of charge. In our part of the galaxy where the primaries have undergone only few collisions before arrival, the electrons are probably those dispersed by the super novae rather than decay electrons of mesons produced in nuclear interactions after the nuclei had escaped into the inter-stellar space. The origin of electrons in the super nova envelope could then be decided by measuring their charge. They will be negative if they bare atomic electrons or result from the β-decay of neutron-rich nuclei. They will be half positive, half negative if they originate in nuclear interactions.

Another important experiment would be to determine the atomic number of the primaries giving rise to extensive air showers. It is known that the composition of the primary cosmic radiation does not depend on energy in the interval extending from less than 1 GeV/nucleon to more than 1 000 GeV/nucleon. Once particles have been accelerated to such an ultra-relativistic energy, it would be difficult to devise a mechanism which can distinguish between particles of different rest mass and discriminate against light or heavy nuclei in further acceleration. It is therefore reasonable to assume that the same composition which one observes at lower energy persists in the energy range where air showers are produced. This means that protons, α-particles, the C, N, O group and the heavier primaries each have a comparable share in producing air showers of a given size.

If now one accepts a theory where the mechanism of acceleration is confined to a restricted space, the acceleration will be terminated when the radius of curvature of the particle trajectories becomes comparable to the dimensions of the region in which the mechanism operates. Then for each component the maximum energy will be that corresponding to this maximum radius of curvature. It follows that the highest part of the primary energy spectrum should be populated entirely by the heaviest nuclei.

It would be important to verify this prediction. If it turns out to be true, then the largest air shower observed so far with an energy a few times 10^{18} eV was produced by a primary particle with magnetic rigidity HR a few times 10^{14} G cm. It seems possible that particles up to such a rigidity can be retained in super nova envelopes. If on the other hand, the largest air showers are to be attributed to protons, then the magnetic rigidity of the primary would have to exceed $7 \cdot 10^{15}$ G cm. The trajectory could not be contained unless there are considerably higher magnetic fields in a super nova envelope than are generally assumed.

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REFERENCES

[1] I. S. ŠKLOVSKIJ: *Dokl. Akad. Nauk, SSSR*, **91** (3), 475 (1953) and other references given in GINZBURG [3].

[2] I. S. ŠKLOVSKIJ: *Suppl. Nuovo Cimento*, **8**, 421 (1958).

[3] V. L. GINZBURG: *Progr. in Elementary Particle and Cosmic Ray Phys.*, vol. **4** (Amsterdam, 1958).

[4] E. FERMI: *Phys. Rev.*, **75**, 1169 (1949).

[5] E. FERMI: *Astrophys. Journ.*, **119**, 1 (1954).

[6] V. L. GINZBURG: *Fortschr. d. Phys.*, **1**, 660 (1954).

[7] B. PETERS: *Progr. in Cosmic Ray Phys.*, vol. **1** (Amsterdam, 1952).

[8] M. F. KAPLON, B. PETERS, H. L. REYNOLDS and D. M. RITSON: *Phys. Rev.*, **85**, 295 (1952).

[9] F. B. MCDONALD: *Phys. Rev.*, **109**, 1367 (1958).

[10] D. LAL: *Proc. Ind. Acad. Sci.*, **38**, 93 (1953).

[11] G. CLARK, J. EARL, W. KRAUSHAAR, J. LINSEY, B. ROSSI and F. SCHERB: *Nature* **180**, 353 (1957)

[12] C. L. CRITCHFIELD, E. P. NEY and S. OLESKA: *Phys. Rev.*, **79**, 402 (1950).

[13] L. I. DORMAN: *Cosmic Ray Variations*, English Translation.

[14] M. V. K. APPA RAO, S. BISWAS, R. R. DANIEL, K. A. NEELAKANTAN and B. PETERS: *Phys. Rev.*, **110**, 751 (1958).

[15] H. TUNMER: *Phil. Mag.*, **3**, 370 (1958).

[16] R. LÜST and J. A SIMPSON: *Phys. Rev.*, **108**, 1563 (1957).

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